Free Oscillations and Rotations of a Rigid Pendulum: Summary of the Theory

1 The Physical System

The most familiar example of a nonlinear mechanical oscillator is an ordinary pendulum in a gravitational field, that is, any rigid body that can swing and rotate about some fixed horizontal axis (a *physical pendulum*), or a massive small bob at the end of a rigid rod of negligible mass (a *simple pendulum*). We employ a rigid rod rather than a flexible string in order to examine complete revolutions of the pendulum as well as its swinging to and fro.

The simple pendulum is a frequently encountered useful physical model. It is interesting not only in itself but more importantly because many problems in the physics of oscillations can be reduced to the differential equation describing the motion of a pendulum.

In the state of stable equilibrium the center of mass of the pendulum is located vertically below the axis of rotation. When the pendulum is deflected from this position through an angle φ, the restoring torque of the gravitational force is proportional to $\sin\varphi$. In the case of small angles φ (i.e., for small oscillations of the pendulum) the values of the sine and of its argument nearly coincide ($\sin\varphi \approx \varphi$), and the pendulum behaves like a linear oscillator. In particular, in the absence of friction it executes *simple harmonic motion*. However, when the amplitude is large, the motion is oscillatory but no longer simple harmonic. In this case, a graph of the angular displacement versus time noticeably departs from a sine curve, and the period of oscillation noticeably depends on the amplitude.

If the angular velocity imparted to the pendulum at its initial excitation is great enough, the pendulum at first executes complete revolutions losing energy through friction, after which it oscillates back and forth.

2 The Differential Equation of Motion for a Pendulum

The equation of rotation of a solid about a fixed axis in the absence of friction in the case of a physical pendulum in a uniform gravitational field is:

$$J\ddot{\varphi} = -mga\sin\varphi. \qquad (1)$$

Here J is the moment of inertia of the pendulum relative the axis of rotation, a is the distance between this axis and the center of mass, and g is the acceleration of gravity. The left-hand side of Eq. (1) is the time rate of change of the angular momentum, and the right-hand side is the restoring torque of the force of gravity. This torque is the product of the force mg (applied at the center of mass) and the lever arm $a \sin\varphi$ of this force. Dividing both sides of Eq. (1) by J we have:

$$\ddot{\varphi} + \omega_0^2 \sin\varphi = 0, \tag{2}$$

where the notation $\omega_0^2 = mga/J$ is introduced.

For a simple pendulum $a = l$, $J = ml^2$, and so $\omega_0^2 = g/l$. For a physical pendulum, the expression for ω_0^2 can be written in the same form as for a simple pendulum provided we define a quantity l to be given by $l = J/(ma)$. It has the dimension of length, and is called the *reduced* or *effective* length of a physical pendulum. Since the differential equation of motion of a physical pendulum with an effective length l is the same as that for a simple pendulum of the same length, the two systems are dynamically equivalent.

3 Physical Parameters of the Pendulum

At small angles of deflection from stable equilibrium, we can replace $\sin\varphi$ with φ in Eq. (2). Then Eq. (2) becomes the differential equation of motion of a linear oscillator. Therefore, the quantity ω_0 in the differential equation of the pendulum, Eq. (2), has the physical sense of the angular frequency of small oscillations of the pendulum in the absence of friction.

In the presence of a torque due to viscous friction, we must add a term to the right-hand side of Eq. (2) which is proportional to the angular velocity $\dot{\varphi}$. Thus, with friction included, the differential equation of the pendulum assumes the form:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2 \sin\varphi = 0. \tag{3}$$

We see that a pendulum is characterized by two parameters: the *angular frequency* ω_0 of small free oscillations, and the *damping constant* γ, which has the dimensions of frequency (or of angular velocity). As in the case of a linear oscillator, it is convenient to use the dimentionless *quality factor* $Q = \omega_0/(2\gamma)$ rather than the damping constant γ to measure the effect of damping. At small free oscillations of the pendulum, the value Q/π is the number of complete cycles during which the amplitude decreases by a factor of $e \approx 2.72$.

The principal difference between Eq. (3) for the pendulum and the corresponding differential equation of motion for a spring oscillator is that Eq. (3)

is a *nonlinear* differential equation. The difficulties in obtaining an analytical solution of Eq. (3) are caused by its nonlinearity. In the general case it is impossible to express the solution of Eq. (3) in elementary functions (although in the absence of friction the solution of Eq. (2) can be given in terms of special functions called elliptic integrals).

4 The Period of Small Oscillations

Nonlinear character of the pendulum is revealed primarily in dependence of the period of oscillations on the amplitude. To find an approximate formula for this dependence, we should keep the next term in the expansion of $\sin\varphi$ in Eq. (2) into the power series:

$$\sin\varphi \approx \varphi - \frac{1}{6}\varphi^3. \tag{4}$$

An approximate solution to the corresponding nonlinear differential equation (for the conservative pendulum with $\gamma = 0$),

$$\ddot{\varphi} + \omega_0^2\varphi - \frac{1}{6}\omega_0^2\varphi^3 = 0, \tag{5}$$

can be searched as a superposition of the sinusoidal oscillation $\varphi(t) = \varphi_{\mathrm{m}}\cos\omega t$ and its third harmonic $\epsilon\varphi_{\mathrm{m}}\cos 3\omega t$ whose frequency equals 3ω. (We assume $t = 0$ to be the moment of maximal deflection). This solution is found in many textbooks. The corresponding derivation is a good exercise for students, allowing them to get an idea of analytical perturbational methods. The fractional contribution ϵ of the third harmonic equals $\varphi_{\mathrm{m}}^2/192$, where φ_{m} is the amplitude of the principal harmonic component whose frequency ω differs from the limiting frequency ω_0 of small oscillations by a term proportional to the square of the amplitude:

$$\omega \approx \omega_0(1 - \varphi_{\mathrm{m}}^2/16), \quad T \approx T_0(1 + \varphi_{\mathrm{m}}^2/16). \tag{6}$$

The same approximate formula for the period, Eq. (6), can be obtained from the exact solution expressed in terms of elliptic integrals by expanding the exact solution into a power series with respect to the amplitude φ_{m}.

Equation (6) shows that, say, for $\varphi_{\mathrm{m}} = 30^\circ$ (0.52 rad) the fractional increment of the period (compared to the period of infinitely small oscillations) equals 0.017 (1.7%). The fractional contribution of the third harmonic in this non-sinusoidal oscillation equals 0.14%, that is, its amplitude equals only 0.043°.

The simulation program allows us to verify this approximate formula for the period. The table below gives the values of T (for several values of the amplitude) calculated with the help of Eq. (6) and measured in the computational experiment. Comparing the values in the last two columns, we see that the approximate formula, Eq. (6), gives the value of the period for the amplitude of 45° with an error of only $(1.0400 - 1.0386)/1.04 = 0.0013 = 0.13\%$. However, for 90° the error is already 2.24%. The error does not exceed 1% for amplitudes up to 70°.

Amplitude φ_{m}		T/T_0 (calculated)	T/T_0 (measured)
30°	$(\pi/6)$	1.0171	1.0175
45°	$(\pi/4)$	1.0386	1.0400
60°	$(\pi/3)$	1.0685	1.0732
90°	$(\pi/2)$	1.1539	1.1803
120°	$(2\pi/3)$	1.2742	1.3730
135°	$(3\pi/4)$	1.3470	1.5279
150°	$(5\pi/6)$	1.4284	1.7622

5 The Phase Portrait of the Pendulum

A general idea about the free motion of the pendulum resulting from various values of energy imparted to the pendulum is given by its phase trajectories. In general, the appearance or structure of a phase diagram tells us a great deal about the possible motions of a nonlinear physical system.

We can construct the family of phase trajectories for a conservative system without explicitly solving the differential equation of motion of the system. The equations for phase trajectories follow directly from the law of the conservation of energy. The potential energy $E_{\mathrm{pot}}(\varphi)$ of a pendulum in the gravitational field depends on the angle of deflection φ measured from the equilibrium position:

$$E_{\mathrm{pot}}(\varphi) = mga(1 - \cos\varphi). \tag{7}$$

A graph of $E_{\mathrm{pot}}(\varphi)$ is shown in the upper part of Fig. 1. The potential energy of the pendulum has a minimal value of zero in the lower stable equilibrium position (at $\varphi = 0$), and a maximal value of $2mga$ (assumed as a unit of energy in Fig. 1) in the inverted position (at $\varphi = \pm\pi$) of unstable equilibrium.

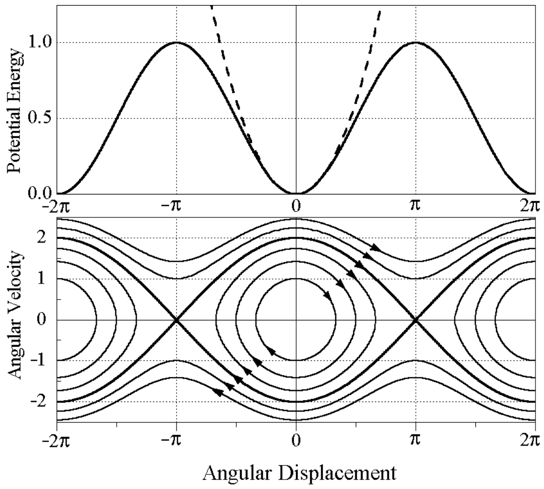

Figure 1: The potential well and the phase portrait of the conservative pendulum.

In the absence of friction, the total energy E of the pendulum, i.e., the sum of its kinetic energy, $\frac{1}{2}J\dot{\varphi}^2$, and potential energy, remains constant during the motion:

$$\frac{1}{2}J\dot{\varphi}^2 + mga(1 - \cos\varphi) = E. \tag{8}$$

This equation gives the relation between $\dot{\varphi}$ and φ, and therefore is the equation of the phase trajectory which corresponds to a definite value E of total energy. It is convenient to express Eq. (8) in a slightly different form. Recalling that $mga/J = \omega_0^2$ and defining the quantity $E_0 = J\omega_0^2/2$ (the quantity E_0 has the physical sense of the kinetic energy of a body with the moment of inertia J, rotating with the angular velocity ω_0), we rewrite Eq. (8):

$$\frac{\dot{\varphi}^2}{\omega_0^2} + 2(1 - \cos\varphi) = \frac{E}{E_0}. \tag{9}$$

If the total energy E of the pendulum is less than the maximal possible value of its potential energy ($E < 2mga = 4E_0$), that is, if the total energy is less than the height of the potential barrier shown in Fig. 1, the pendulum swings back and forth between the extreme deflections φ_m and $-\varphi_m$. These angles correspond to the extreme points at which the potential energy $E_{\mathrm{pot}}(\varphi)$ becomes equal to the total energy E of the pendulum. If the amplitude is small ($\varphi_m \ll \pi/2$), the oscillations are nearly sinusoidal in time, and the corresponding phase trajectory is nearly an ellipse. The elliptical shape of the curve follows from Eq. (9) if we substitute there the approximate expression $\cos\varphi \approx 1 - \varphi^2/2$ valid for small angles φ:

$$\frac{\dot{\varphi}^2}{E\omega_0^2/E_0} + \frac{\varphi^2}{E/E_0} = 1. \tag{10}$$

This is the equation of an ellipse in the phase plane $(\varphi,\ \dot{\varphi})$. Its horizontal semiaxis equals the maximal deflection angle $\varphi_m = \sqrt{E/E_0}$. If the angular velocity $\dot{\varphi}$ on the ordinate axis is plotted in units of the angular frequency ω_0 of small free oscillations, the ellipse (10) becomes a circle.

The greater the total energy E (and thus the greater the amplitude φ_m), the greater the departure of the motion from simple harmonic and the greater the departure of the phase trajectory from an ellipse. The width of the phase trajectory increases horizontally (along φ-axis) as the energy E increases to $2mga$.

If $E > 2mga$, the kinetic energy and the angular velocity of the pendulum are non-zero even at $\varphi = \pm\pi$. In contrast to the case of swinging, now the angular velocity does not change its sign. The pendulum executes *rotation* in a full circle. This rotation is nonuniform. When the pendulum passes through the lowest point (through the position of stable equilibrium), its angular velocity is greatest, and when the pendulum passes through the highest point (through the position of unstable equilibrium), its angular velocity is smallest.

In the phase plane, rotation of the pendulum is represented by the paths which continue beyond the vertical lines $\varphi = \pm\pi$, repeating themselves every full cycle of revolution, as shown in Fig. 1. Upper paths lying above the φ-axis, where $\dot{\varphi}$ is positive and φ grows in value, correspond to counterclockwise rotation, and paths below the axis, along which the representative point moves from the right to the left, correspond to clockwise rotation of the pendulum.

For a conservative system, the equation of a phase trajectory (e.g., Eq. (9) in the case of a pendulum) is always an even function of $\dot{\varphi}$, because the energy depends only on $\dot{\varphi}^2$. Consequently, the phase trajectory of a conservative system is symmetric about the horizontal φ-axis. This symmetry means that the motion of the system in the clockwise direction is mechanically the same as the motion in the counterclockwise direction. In other words, the motion of a conservative system is *reversible*: if we instantaneously change the sign of its velocity, the representative point jumps to the symmetric position of the same phase trajectory on the other side of the horizontal φ-axis. In the reverse motion the system passes through each spatial point φ with the same speed as in the direct motion. Since changing the sign of the velocity ($\dot{\varphi} \to -\dot{\varphi}$) is the same as changing the sign of time ($t \to -t$), this property of a conservative system is also referred to as the *symmetry of time reversal.*

The additional symmetry of the phase trajectories of the conservative pendulum about the vertical $\dot{\varphi}$-axis (with respect to the change $\varphi \to -\varphi$) follows from the symmetry of its potential well: $E_{\text{pot}}(-\varphi) = E_{\text{pot}}(\varphi)$. (Unlike the symmetry about the φ-axis, this additional symmetry is not a property of all conservative systems.)

When we include friction in our model, motion of the pendulum becomes irreversible, and the discussed above symmetry of its phase trajectories with respect to reflections in the coordinate axes of the phase plane vanishes. The influence of friction on the phase portrait we discuss below (section 7).

The angles φ and $\varphi \pm 2\pi$, $\varphi \pm 4\pi$, ... denote the same position of the pendulum and thus are equivalent (the angle of deflection φ is a *cyclic* variable). Thus it is sufficient to consider only a part of the phase plane, e.g., the part enclosed between the vertical lines $\varphi = -\pi$ and $\varphi = \pi$ (see Fig. 1). The cyclic motion of the pendulum in the phase plane is then restricted to the region lying between these vertical lines. We can identify these lines and assume that when the representative point leaves the region crossing the right boundary $\varphi = \pi$, it enters simultaneously from the opposite side at the left boundary $\varphi = -\pi$ (for a counterclockwise rotation of the pendulum).

We can imagine the two-dimensional phase space of a rigid pendulum not only as a part of the plane (φ, $\dot{\varphi}$) enclosed between the vertical lines $\varphi = +\pi$ and $\varphi = -\pi$, but also as a continuous surface. We may do so because opposing points on these vertical lines have the same value of $\dot{\varphi}$ and describe physically equivalent mechanical states. And so, taking into account the identity of the mechanical states of the pendulum at these points and the periodicity of the dependence of the restoring gravitational torque on φ, we can cut out this part of the phase plane and roll it into a cylinder so that the bounding lines $\varphi = +\pi$ and $\varphi = -\pi$ are joined. We can thus consider the surface of such a cylinder as the phase space of a rigid pendulum. A phase curve circling around the cylinder corresponds to a nonuniform rotational motion of the pendulum.

6 Limiting Motion along the Separatrix

The phase trajectory corresponding to a total energy E which is equal to the maximal possible potential energy, namely $E_{\text{pot}}(\pi) = 2mga$, is of special interest. It separates the central region of the phase plane which is occupied by the closed phase trajectories of oscillations from the outer region, occupied by the phase trajectories of rotations. This boundary is called the *separatrix*. In Fig. 1 it is singled out by a thick line. The separatrix divides the phase plane of a conservative pendulum into regions which correspond to different types of motion. The equation of the separatrix follows from Eq. (8) by setting $E = 2mga$, or from Eq. (9) by setting $E = 4E_0 = 2J\omega_0^2$:

$$\dot{\varphi} = \pm 2\omega_0 \cos(\varphi/2). \tag{11}$$

The limiting motion of a conservative pendulum with total energy $E = 2mga$ is worth a more detailed investigation. In this case the representative point in the phase plane moves along the separatrix.

When the pendulum with the energy $E = 2mga$ approaches the inverted position at $\varphi = \pi$ or $\varphi = -\pi$, its velocity approaches zero, becoming zero at $\varphi = \pm\pi$. This state is represented in the phase plane by the saddle points $\varphi = \pi$, $\dot{\varphi} = 0$ and $\varphi = -\pi$, $\dot{\varphi} = 0$ where the upper and lower branches of the separatrix, Eq. (11), meet on the φ-axis. Both these points represent the same mechanical state of the system, that in which the pendulum is at rest in the unstable inverted position. The slightest initial displacement of the pendulum from this point to one side or the other results in its swinging with an amplitude which almost equals π, and the slightest initial push causes rotational motion (revolution) of the pendulum in a full circle. With such swinging, or with such rotatation, the pendulum remains in the vicinity of the inverted position for an extended time.

For the case of motion along the separatrix, i.e., for the motion of the pendulum with total energy $E = 2mga = 4E_0$, there exist an analytical solution (in elementary functions) for the angle of deflection $\varphi(t)$ and for the angular velocity $\dot{\varphi}(t)$. Integration of the differential equation Eq. (11) with respect to time (for the positive sign of the root) at the initial condition $\varphi(0) = 0$ yields:

$$-\,\omega_0 t = \ln\, \tan[(\pi - \varphi)/4], \tag{12}$$

and we obtain the following expression for $\varphi(t)$:

$$\varphi(t) = \pi - 4\arctan(e^{-\omega_0 t}). \tag{13}$$

This solution describes a counterclockwise motion beginning at $t = -\infty$ from $\varphi = -\pi$. At $t = 0$ the pendulum passes through the bottom of its circular path, and continues its motion until $t = +\infty$, asymptotically approaching $\varphi = +\pi$. A graph of $\varphi(t)$ for this motion is shown in Fig. 2.

The second solution which corresponds to the clockwise motion of the pendulum (to the motion along the other branch of the separatrix in the

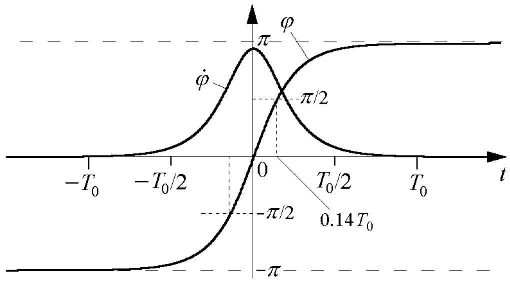

Figure 2: The graphs of φ and $\dot{\varphi}$ for the limiting motion (total energy $E = 2mga = 4E_0$).

phase plane) can be obtained from Eq. (13) by the transformation of time reversal, i.e., by the change $t \to -t$. Solutions with different initial conditions can be obtained from Eq. (13) simply by a shift of the time origin (by the substitution of $t - t_0$ for t).

To obtain the time dependence of the angular velocity $\dot{\varphi}(t)$ for the limiting motion of the pendulum, we can express $\cos(\varphi/2)$ from Eq. (13) as:

$$\cos(\varphi/2) = \frac{1}{\cosh(\omega_0 t)},$$

and after substitution of this value into Eq. (11), we find:

$$\dot{\varphi}(t) = \pm\frac{2\omega_0}{\cosh(\omega_0 t)} = \pm\frac{4\omega_0}{(e^{\omega_0 t} + e^{-\omega_0 t})}. \tag{14}$$

A graph of $\dot{\varphi}(t)$ is also shown in Fig. 2. The graph of this function has the form of an isolated impulse. The characteristic width of its profile, i.e., the duration of such a solitary impulse, is of the order of $1/\omega_0$. Consequently, the time needed for the pendulum to execute almost all of its circular path, from the vicinity of the inverted position through the lowest point and back, has the order of magnitude of one period of small free oscillations, $T = 2\pi/\omega_0$. The wings of the profile decrease exponentially as $t \to \pm\infty$. Actually, for large positive values of t, we may neglect the second term $\exp(-\omega_0 t)$ in the denominator of Eq. (14), and we find that:

$$\dot{\varphi}(t) \approx \pm 4\omega_0 e^{-\omega_0 t}. \tag{15}$$

Thus, in the limiting motion of the representative point along the separatrix, when the total energy E is exactly equal to the height $2mga$ of the potential barrier, the speed of the pendulum decreases steadily as it nears the inverted position of unstable equilibrium. The pendulum approaches the inverted position asymptotically, requiring an infinite time to reach it.

The mathematical relationships associated with the limiting motion of a pendulum along the separatrix play an important role in the theory of solitons.

7 Period of Large Oscillations and Revolutions

If the energy differs from the critical value $2mga$, the motion of the pendulum in the absence of friction (swinging at $E < 2mga$ or rotation at $E > 2mga$) is periodic. The period T of such a motion the greater the closer the energy E to $2mga$. Figure 3 gives the dependence of the period on the total energy $T(E)$. (The energy is measured in units of the maximal potential energy $2mga$.)

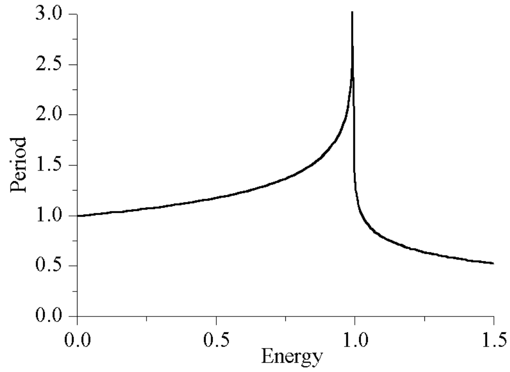

Figure 3: The period versus total energy.

The initial almost linear growth of the period with E corresponds to the approximate formula, Eq. (6). Indeed, Eq. (6) predicts a linear dependence of T on φ_{m}^2, and for small amplitudes φ_{m} the energy is proportional to the square of the amplitude. When the energy approaches the value $2mga$, the period grows infinitely. Greater values of the energy correspond to the rotating pendulum. The period of rotation decreases with the energy. The asymptotic behavior of the period at $E \gg 2mga$ can be found as follows.

When the total energy E of the pendulum is considerably greater than the maximal value $2mga$ of its potential energy, we can assume all the energy of the pendulum to be the kinetic energy of its rotation. In other words, we can neglect the influence of the gravitational field on the rotation and consider this rotation to be uniform. The angular velocity of this rotation is approximately equal to the angular velocity Ω received by the pendulum at the initial excitation. The period T of rotation is inversely proportional to the angular velocity of rotation: $T = 2\pi/\Omega$. So for $E = J\Omega^2/2 \gg 2mga$ the asymptotic dependence of the period on the initial angular velocity is the inverse proportion: $T \propto 1/\Omega$.

To find the dependence $T(\Omega)$ more precisely, we need to take into account the variations in the angular velocity caused by gravitation. The angular velocity of the pendulum oscillates between the maximal value Ω in the lower position and the minimal value $\Omega_{\min}$ in the upper position. The latter can be found from the conservation of energy:

$$\Omega_{\min} = \sqrt{\Omega^2 - 4\omega_0^2} \approx \Omega\left(1 - 2\frac{\omega_0^2}{\Omega^2}\right).$$

For rapid rotation we can assume these oscillations of the angular velocity to be almost sinusoidal. Then the average angular velocity of rotation is approximately the half-sum of its maximal and minimal values:

$$\Omega_{\mathrm{av}} \approx (\Omega + \Omega_{\min})/2 = \Omega(1 - \omega_0^2/\Omega^2),$$

and the period of rotation is:

$$T(\Omega) = \frac{2\pi}{\Omega_{\mathrm{av}}} \approx T_0\,\frac{\omega_0}{\Omega}\left(1 + \frac{\omega_0^2}{\Omega^2}\right).$$

However, the most interesting peculiarities are revealed if we investigate the dependence of the period on energy in the vicinity of $E_{\max} = 2mga$.

Measuring the period of oscillations for the amplitudes 179.900°, 179.990°, and 179.999° successively, we see that duration of the impulses on the graph of the angular velocity very nearly remains the same, but the intervals between them become longer as the amplitude approaches 180°: Experimental values of the period T of such extraordinary oscillations are respectively $5.5\,T_0$, $6.8\,T_0$, and $8.3\,T_0$.

It is interesting to compare the motions for two values of the total energy E which differ slightly from $E_{\max}$ on either side by the same amount, i.e., for $E/E_{\max} = 0.9999$ and $E/E_{\max} = 1.0001$. In the phase plane, these motions occur very near to the separatrix, the first one inside and the latter outside of the separatrix. The inner closed curve corresponds to oscillations with the amplitude 178.9°. Measuring the periods of these motions, we obtain the values $3.814\,T_0$ and $1.907\,T_0$ respectively. That is, the measured period of these oscillations is exactly twice the period of rotation, whose phase curve adjoins the separatrix from the outside.

The graphs of $\varphi(t)$ and $\dot{\varphi}(t)$ for oscillations and revolutions of the pendulum whose energy equals $E_{\max} \mp \Delta E$ are shown respectively in the upper and lower parts of Fig. 4.

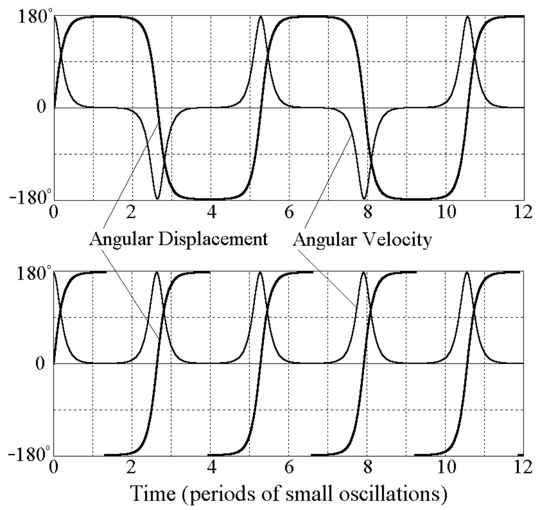

Figure 4: The graphs of $\varphi(t)$ and $\dot{\varphi}(t)$ for the pendulum excited at $\varphi = 0$ by imparting the initial angular velocity of $\dot{\varphi} = 2\omega_0(1 \mp 10^{-6})$.

Next we suggest a theoretical approach which can be used to calculate the period of oscillations and revolutions with $E \approx 2mga$.

From the simulation experiments we can conclude that during the semicircular path, from the equilibrium position up to the extreme deflection or to the inverted position, both of the motions shown in Fig. 4 almost coincide with the limiting motion (Fig. 2). These motions differ from the limiting motion appreciably only in the immediate vicinity of the extreme point or near the inverted position: In the first case the pendulum stops at this extreme point and then begins to move backwards, while in the limiting motion the pendulum continues moving for an unlimited time towards the inverted position; in the second case the pendulum reaches the inverted position during a finite time.

For the oscillatory motion under consideration, the representative point in the phase plane generates a closed path during one cycle, passing along both branches of the separatrix. In this motion the pendulum goes twice around almost the whole circle, covering it in both directions. On the other hand, executing rotation, the pendulum makes one circle during a cycle of revolutions, and the representative point passes along one branch of the separatrix (upper or lower, depending on the direction of rotation). To explain why the period of these oscillations is twice the period of corresponding revolutions, we must show that the motion of the pendulum with energy $E = 2mga - \Delta E$ from $\varphi = 0$ up to the extreme point requires the same time as the motion with the energy $E = 2mga + \Delta E$ from $\varphi = 0$ up to the inverted vertical position.

Almost all of both motions occurs very nearly along the same path in the phase plane, namely, along the separatrix from the initial point $\varphi = 0$, $\dot{\varphi} \approx 2\omega_0$ up to some angle φ_0 whose value is close to π. To calculate the time interval required for this part of the motion, we can assume that the motion (in both cases) occurs exactly along the separatrix, and take advantage of the corresponding analytical solution, expressed by Eq. (13).

Assuming $\varphi(t)$ in Eq. (13) to be equal to φ_0, we can find the time t_0 during which the pendulum moves from the equilibrium position $\varphi = 0$ up to the angle φ_0 (for both cases):

$$\omega_0 t_0 = -\ln\tan\frac{\pi - \varphi_0}{4} = -\ln\tan\frac{\alpha_0}{4}, \tag{16}$$

where we have introduced the notation $\alpha_0 = \pi - \varphi_0$ for the angle that the rod of the pendulum at $\varphi = \varphi_0$ forms with the upward vertical line. When φ_0 is close to π, the angle α_0 is small, so that in Eq. (16) we can assume $\tan(\alpha_0/4) \approx \alpha_0/4$, and $\omega_0 t_0 \approx \ln(4/\alpha_0)$.

Later we shall consider in detail the subsequent part of motion which occurs from this arbitrarily chosen angle $\varphi = \varphi_0$ towards the inverted position, and prove that the time t_1 required for the pendulum with the energy $2mga + \Delta E$ (rotational motion) to reach the inverted position $\varphi = \pi$ equals the time t_2 during which the pendulum with the energy $2mga - \Delta E$ (oscillatory motion) moves from φ_0 up to its extreme deflection φ_{m}, where the angular velocity becomes zero, and the pendulum begins to move backwards.

When considering the motion of the pendulum in the vicinity of the inverted position, we find it convenient to define its position (instead of the angle φ) by the angle α of deflection from this position of unstable equilibrium. This angle equals $\pi - \varphi$, and the angular velocity $\dot{\alpha}$ equals $-\dot{\varphi}$. The potential energy of the pendulum (measured relative to the lower equilibrium position) depends on α in the following way:

$$E_{\mathrm{pot}}(\alpha) = mga(1 + \cos\alpha) \approx 2mga(1 - \frac{1}{4}\alpha^2). \tag{17}$$

The latter expression is valid only for relatively small values of α, when the pendulum moves near the inverted position. Phase trajectories of motion with energies $E = 2mga \pm \Delta E$ near the saddle point $\varphi = \pi$, $\dot{\varphi} = 0$ (in the new variables $\alpha = 0, \dot{\alpha} = 0$) can be found from the conservation of energy using the approximate expression (17) for the potential energy:

$$\frac{1}{2}J\dot{\alpha}^2 + \frac{1}{2}mga\alpha^2 = \pm\Delta E, \text{ or } \frac{\dot{\alpha}^2}{\omega_0^2} - \alpha^2 = \pm 4\varepsilon. \tag{18}$$

Here we use the notation $\varepsilon = \Delta E/E_{\max} = \Delta E/(2mga)$ for the small ($\varepsilon \ll 1$) dimensionless quantity characterizing the fractional deviation of energy E from its value $E_{\max}$ for the separatrix. It follows from Eq. (18) that phase trajectories near the saddle point are hyperbolas whose asymptotas are the two branches of the separatrix that meet at the saddle point. Part of the phase portrait near the saddle point is shown in Figure 5. The curve *1* for the energy $E = 2mga + \Delta E$ corresponds to the rotation of the pendulum. It intersects the ordinate axis when the pendulum passes through the inverted position. The curve *2* for the energy $E = 2mga - \Delta E$ describes the oscillatory motion. It intersects the abscissa axis at the distance $\alpha_{\mathrm{m}} = \pi - \varphi_{\mathrm{m}}$ from the origin. This is the point of extreme deflection in the oscillations.

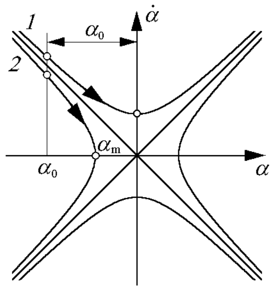

Figure 5: Phase curves near the saddle point.

For $\alpha \ll 1$ the torque of the gravitational force is approximately proportional to the angle α, but in contrast to the case of stable equilibrium, the torque $N = -dE_{\mathrm{pot}}(\alpha)/d\alpha = mga\alpha$ tends to move the pendulum farther from the position $\alpha = 0$ of unstable equilibrium. Substituting the torque N in the law of rotation of a solid, we find the differential equation of the pendulum valid for its motion near the point $\alpha = 0$:

$$J\,\ddot{\alpha} = mga\,\alpha, \quad \text{or} \quad \ddot{\alpha} - \omega_0^2\alpha = 0. \tag{19}$$

The general solution of this linear equation can be represented as a superposition of two exponential functions of time t:

$$\alpha(t) = C_1 e^{\omega_0 t} + C_2 e^{-\omega_0 t}. \tag{20}$$

Next we consider separately the two cases of motion with the energies $E = E_{\max} \pm \Delta E$.

1. Rotational motion ($E = E_{\max} + \Delta E$) along the curve *1* from α_0 up to the intersection with the ordinate axis. Let $t = 0$ be the moment of crossing the inverted vertical position: $\alpha(0) = 0$. Hence in Eq. (20) $C_2 = -C_1$. Then from Eq. (18) $\dot{\alpha}(0) = 2\sqrt{\varepsilon}\omega_0$, and $C_1 = \sqrt{\varepsilon}$. To determine duration t_1 of the motion, we assume in Eq. (20) $\alpha(t_1) = \alpha_0$:

$$\alpha_0 = \sqrt{\varepsilon}(e^{\omega_0 t_1} - e^{-\omega_0 t_1}) \approx \sqrt{\varepsilon} e^{\omega_0 t_1}.$$

(We can choose here an arbitrary value α_0, although a small one, to be large compared to $\sqrt{\varepsilon}$, so that the condition $e^{-\omega_0 t_1} \ll e^{\omega_0 t_1}$ is fulfilled). Therefore $\omega_0 t_1 = \ln(\alpha_0/\sqrt{\varepsilon})$.

2. Oscillatory motion ($E = E_{\max} - \Delta E$) along the curve *2* from α_0 up to the extreme point α_{m}. Let $t = 0$ be the moment of maximal deflection, when the phase curve intersects the abscissa axis: $\dot{\alpha}(0) = 0$. Hence in Eq. (20) $C_2 = C_1$. Then from Eq. (18) $\alpha(0) = \alpha_{\mathrm{m}} = 2\sqrt{\varepsilon}$, and $C_1 = \sqrt{\varepsilon}$. To determine duration t_2 of this motion, we assume in Eq. (20) $\alpha(t_2) = \alpha_0$. Hence

$$\alpha_0 = \sqrt{\varepsilon}(e^{\omega_0 t_2} + e^{-\omega_0 t_2}) \approx \sqrt{\varepsilon} e^{\omega_0 t_2},$$

and we find $\omega_0 t_2 = \ln(\alpha_0/\sqrt{\varepsilon})$.

We see that $t_2 = t_1$ if $\varepsilon = \Delta E/E_{\max}$ is the same in both cases. Therefore the period of oscillations is twice the period of rotation for the values of energy which differ from the critical value $2mga$ on both sides by the same small amount ΔE. Indeed, we can assume with great precision that the motion from $\varphi = 0$ up to $\varphi_0 = \pi - \alpha_0$ lasts the same time t_0 given by Eq. (16), since these parts of both phase trajectories very nearly coincide with the separatrix. In the case of rotation, the remaining motion from φ_0 up to the inverted position also lasts the same time as, in the case of oscillations, does the motion from φ_0 up to the utmost deflection φ_{m}, since $t_1 = t_2$.

The period of rotation T_{rot} is twice the duration $t_0 + t_1$ of motion from the equilibrium position $\varphi = 0$ up to the $\varphi = \pi$. Using the above value for t_1 and Eq. (16) for t_0, we find:

$$T_{\mathrm{rot}} = 2(t_0 + t_1) = \frac{2}{\omega_0} \ln \frac{4}{\sqrt{\varepsilon}} = \frac{1}{\pi} T_0 \ln \frac{4}{\sqrt{\varepsilon}}.$$

We note that an arbitrarily chosen angle α_0 (however, $\sqrt{\varepsilon} \ll \alpha_0 \ll 1$), which delimits the two stages of motion (along the separatrix, and near the saddle point in the phase plane), falls out of the final formula for the period (when we add t_0 and t_1). The period of revolutions tends to infinity when $\varepsilon \to 0$, that is, when the energy tends to its critical value $2mga$. For $\varepsilon = 0.0001$ (for $E = 1.0001E_{\text{max}}$) the above formula gives the value $T_{\text{rot}} = 1.907\,T_0$, which coincides with the cited experimental result.

The period of oscillations T is four times greater than the duration t_0+t_2 of motion from $\varphi = 0$ up to the extreme point φ_{m}:

$$T = 4(t_0 + t_2) = \frac{4}{\omega_0}\ln\frac{4}{\sqrt{\varepsilon}} = \frac{2}{\pi}T_0\ln\frac{8}{\alpha_{\text{m}}}.$$

For $\alpha_{\text{m}} \ll 1$ ($\varphi_{\text{m}} \approx \pi$) this formula agrees well with the experimental results: it yields $T = 5.37\,T_0$ for $\varphi_{\text{m}} = 179.900°$, $T = 6.83\,T_0$ for $\varphi_{\text{m}} = 179.990°$, and $T = 8.30\,T_0$ for $\varphi_{\text{m}} = 179.999°$. From the obtained expressions we see how both the period of oscillations T and the period of rotation T_{rot} tend to infinity as the total energy approaches $E_{\text{max}} = 2mga$.

As the total energy is changed so as to approach the value $2mga$ from below, the period of oscillations sharply increases and tends logarithmically to infinity. The shape of the curve of angular velocity versus time resembles a periodic succession of solitary impulses whose duration is close to the period T_0 of small oscillations (see Fig. 4.). Time intervals between successive impulses are considerably greater than T_0.

These intervals grow longer and longer as the total energy E is changed so as to approach the maximal allowed potential energy $2mga$. The phase trajectory of the motion becomes closer and closer to the separatrix from the inside. Executing such swinging, the pendulum moves very rapidly through the bottom of its circular path and very slowly at the top, in the vicinity of the extreme points.

As the extreme angular displacement approaches 180°, the pendulum spends the greater part of its period near the inverted position, and so the potential energy of the pendulum is close to its maximal value $2mga$ most of the time. Only for the brief time during which the pendulum rotates rapidly through the bottom part of its circular path is the potential energy converted into kinetic energy. Hence the time average value of the potential energy during a complete cycle of this motion is considerably greater than the time average value of the kinetic energy, in contrast to the case of small oscillations, for which the time average values of potential and kinetic energies are equal.

8 The Mean Energies

In the motion under consideration (swinging or rotation with $E \approx 2mga$) both potential and kinetic energies oscillate between zero and the same maximal value, which is equal to the total energy $E \approx 2mga$. However, during almost all the period the pendulum moves very slowly in the vicinity of the inverted position, and during this time its potential energy has almost the maximal value $2mga = 2J\omega_0^2$. Only for a short time, when the pendulum passes rapidly along the circle and through the bottom of the potential well, is the potential energy of the pendulum converted into kinetic energy. Hence, on the average, the potential energy predominates.

We can estimate the ratio of the averaged over a period values of the potential and kinetic energies if we take into account that most of the time the angular velocity of the pendulum is nearly zero, and for a brief time of motion the time dependence of $\varphi(t)$ is very nearly the same as it is for the limiting motion along the separatrix. Therefore we can assume that during an impulse the kinetic energy depends on time in the same way it does in the limiting motion. This assumption allows us to extend the limits of integration to $\pm\infty$. Since two sharp impulses of the angular velocity (and of the kinetic energy) occur during the period T of oscillations, we can write:

$$\langle E_{\mathrm{kin}} \rangle = \frac{J}{T}\int_{-\infty}^{\infty} \dot{\varphi}^2(t)dt = \frac{J}{T}\int_{-\pi}^{\pi} \dot{\varphi}(\varphi)d\varphi.$$

The integration with respect to time is replaced here with an integration over the angle. The mean kinetic energy $\langle E_{\mathrm{kin}} \rangle$ is proportional to the area S of the phase plane bounded by the separatrix: $2\langle E_{\mathrm{kin}} \rangle = JS/T$. We can substitute for $\dot{\varphi}(\varphi)$ its expression from the equation of the separatrix, Eq. (11):

$$\langle E_{\mathrm{kin}} \rangle = \frac{J}{T}2\omega_0\int_{-\pi}^{\pi} \cos\frac{\varphi}{2}d\varphi = \frac{4}{\pi}J\omega_0^2\frac{T_0}{T}.$$

Taking into account that the total energy E for this motion approximately equals $2mga = 2J\omega_0^2$, and $E_{\mathrm{pot}} = E - E_{\mathrm{kin}}$, we find:

$$\frac{\langle E_{\mathrm{pot}} \rangle}{\langle E_{\mathrm{kin}} \rangle} = \frac{2J\omega_0^2}{\langle E_{\mathrm{kin}} \rangle} - 1 = \frac{\pi}{2}\frac{T}{T_0} - 1.$$

For $\varphi_{\mathrm{m}} = 179.99^\circ$ the period T equals $6.83\,T_0$, and so the ratio of mean values of potential and kinetic energies is 9.7 (compare with the case of small oscillations for which these mean values are equal).

9 The Influence of Friction

In the presence of weak friction inevitable in any real system, the phase portrait of the pendulum changes qualitatively: The phase curves have a different topology (compare Figures 6 and 1). A phase trajectory representing the counterclockwise rotation of the pendulum sinks lower and lower toward the separatrix with each revolution. The phase curve which passed formerly along the upper branch of the separatrix does not reach now the saddle point $(\pi,\ 0)$. Instead it begins to wind around the origin, gradually approaching it. Similarly, the lower branch crosses the abscissa axis $\dot{\varphi} = 0$ to the right of the saddle point $(-\pi,\ 0)$, and also spirals towards the origin.

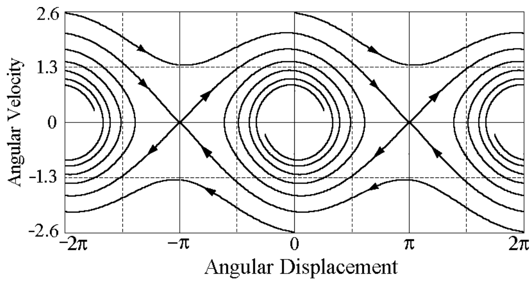

Figure 6: Phase portrait with friction.

The closed phase trajectories corresponding to oscillations of a conservative system are transformed by friction into shrinking spirals which wind around a *focus* located at the origin of the phase plane. Near the focus the size of gradually shrinking loops diminishes in a geometric progression. This focus represents a state of rest in the equilibrium position, and is an *attractor* of the phase trajectories. That is, all phase trajectories of the damped pendulum spiral in toward the focus, forming an infinite number of loops, as in Figure 7, *a*.

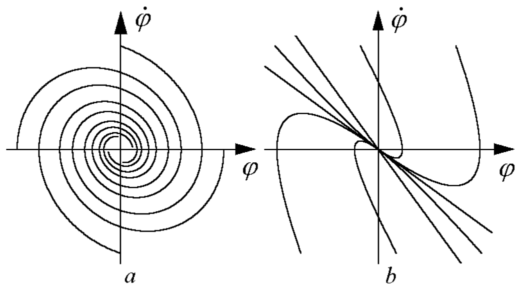

Figure 7: Phase portrait of a damped (a) and of an overdamped ($\gamma > \omega_0$, b) pendulum.

When friction is relatively strong ($\gamma > \omega_0$), the motion is non-oscillatory, and the attractor of the phase trajectories, instead of a focus, becomes a *node*: all phase trajectories approach this node directly, without spiraling. The phase portrait of an overdamped pendulum ($\gamma = 1.05\,\omega_0$) is shown in Figure 7,*b*.

When friction is weak, we can make some theoretical predictions for the motions whose phase trajectories pass close to the separatrix. For example, we can evaluate the minimal value of the initial velocity which the pendulum must be given in the lower (or some other) initial position in order to reach the inverted position, assuming that the motion occurs along the separatrix, and consequently that the dependence of the angular velocity on the angle of deflection is approximately given by the equation of the separatrix, Eq. (11).

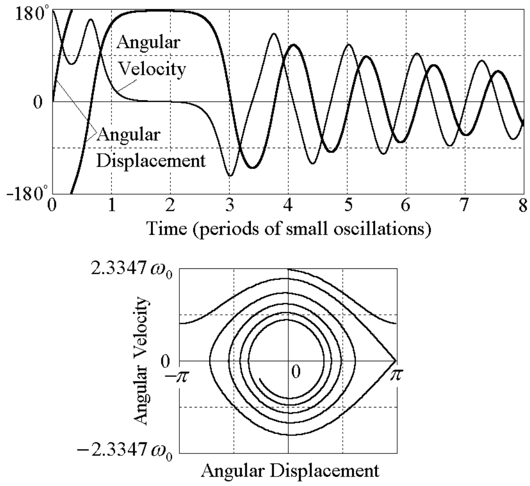

Figure 8: Revolution and subsequent oscillation of the pendulum with friction ($Q = 20$) excited from the equilibrium position with an initial angular velocity of $\Omega = 2.3347\,\omega_0$.

The frictional torque is proportional to the angular velocity: $N_{\rm fr} = -2\gamma J\dot{\varphi}$. Substituting the angular velocity from Eq. (11), we find

$$N_{\rm fr} = \mp 4\gamma J\omega_0 \cos\frac{\varphi}{2} = \mp\frac{2mga}{Q}\cos\frac{\varphi}{2}.$$

Hence the work $W_{\rm fr}$ of the frictional force during the motion from an initial point φ_0 to the final inverted position $\varphi = \pm\pi$ is:

$$W_{\rm fr} = \int_{\varphi_0}^{\pm\pi} N_{\rm fr}d\varphi = -4\frac{mga}{Q}\left(1 \mp \sin\frac{\varphi_0}{2}\right). \qquad (21)$$

The necessary value of the initial angular velocity Ω can be found with the help of the conservation of energy, in which the work $W_{\rm fr}$ of the frictional force is taken into account:

$$\frac{1}{2}J\Omega^2 + mga(1 - \cos\varphi_0) + W_{\rm fr} = 2mga.$$

Substituting $W_{\rm fr}$ from Eq. (21), we obtain the following expression for Ω:

$$\Omega^2 = 2\omega_0\left[1 + \cos\varphi_0 + \frac{4}{Q}\left(1 \mp \sin\frac{\varphi_0}{2}\right)\right]. \qquad (22)$$

For $\varphi_0 \neq 0$ the sign in Eq. (22) depends on direction of the initial angular velocity. We must take the upper sign if the pendulum moves directly to the inverted position, and the lower sign if it passes first through the lower equilibrium position. In other words, at $\varphi_0 > 0$ we must take the upper sign for positive values of Ω, and the lower sign otherwise. If the pendulum is excited from the lower equilibrium position ($\varphi_0 = 0$), Eq. (22) yields the initial velocity to be

$$\Omega = \pm 2\omega_0\sqrt{1 + 2/Q} \approx \pm 2\omega_0(1 + 1/Q).$$

The exact value of Ω is slightly greater since the motion towards the inverted position occurs in the phase plane close to the separatrix but always outside it, that is, with the angular velocity of slightly greater magnitude. Consequently, the work $W_{\rm fr}$ of the frictional force during this motion is a little larger than the calculated value. For example, for $Q = 20$ the above estimate yields $\Omega = \pm 2.1\omega_0$, while the more precise value of Ω determined experimentally by trial and error is $\pm 2.10096\,\omega_0$.

Figure 8 shows the graphs of $\varphi(t)$ and $\dot\varphi(t)$ and the phase trajectory for a similar case in which the initial angular velocity is chosen exactly to let the pendulum reach the inverted position after a revolution.

Supplement: Review of the Principal Formulas

The differential equation of motion of a rigid pendulum is:

$$\ddot\varphi + 2\gamma\dot\varphi + \omega_0^2\sin\varphi = 0,$$

where ω_0 is the frequency of small free oscillations:

$$\omega_0^2 = mga/J = g/l; \qquad l = J/ma.$$

Here m is the mass of the pendulum, a is the distance between the horizontal axis of rotation (the point of suspension) and the center of mass, J is the moment of inertia about the same axis, l is the reduced length of the physical pendulum, and g is the acceleration of gravity.

The equation of a phase trajectory in the absence of friction is:

$$\frac{\dot{\varphi}^2}{\omega_0^2} + 2(1 - \cos\varphi) = \frac{E}{E_0},$$

where E is the total energy, and

$$E_0 = \frac{1}{2}J\omega_0^2 = \frac{1}{2}mga = \frac{1}{4}(E_{\mathrm{pot}})_{\mathrm{max}}.$$

Here $(E_{\mathrm{pot}})_{\mathrm{max}} = 2mga$ is the maximal possible value of the potential energy of the pendulum, which is its potential energy when it is in the inverted vertical position.

The equation of the separatrix in the phase plane is:

$$\dot{\varphi} = \pm 2\omega_0 \cos(\varphi/2).$$

The angular deflection and angular velocity for the motion of the pendulum which generates the separatrix in the phase plane are:

$$\varphi(t) = \pi - 4\arctan(e^{-\omega_0 t}), \qquad \dot{\varphi}(t) = \pm\frac{2\omega_0}{\cosh(\omega_0 t)}. = \pm\frac{4\omega_0}{(e^{\omega_0 t} + e^{-\omega_0 t})}.$$