Square-wave External Force in a Linear System

Manual

Eugene Butikov

Annotation. The manual includes a description of the simulated physical system and a summary of the relevant theoretical material for students as a prerequisite for the virtual lab “Square-wave Excitation of Linear Torsion Pendulum.” The manual includes also a set of theoretical and experimental problems to be solved by students on their own, as well as various assignments which the instructor can offer students for possible individual mini-research projects.

Contents

1 Summary of the Theory

This virtual lab deals with oscillations of a torsion spring pendulum excited by abrupt periodic displacements of its equilibrium position, or, which is equivalent, with forced oscillations under an external periodic square-wave force.

1.1 The Physical System

To study forced oscillations caused by a non-sinusoidal external influence, we employ here the same mechanical device as in the simulations of free oscillations and of oscillations forced by a harmonic external torque—a balanced rotor (flywheel) attached to one end of a spiral spring. A schematic image of the torsion spring oscillator as it is displayed on the computer screen is shown in figure 1. The flywheel turns about its axis under the restoring torque of the spring, much like the balance devices used in mechanical watches. The other end of the spiral spring is attached to a driving rod which can be turned about the axis common with the axis of the flywheel. When this rod is turned, the equilibrium position of the flywheel is displaced alongside the rod through the same angle. The flywheel can execute damped free oscillations about this new displaced equilibrium position. For weak and moderate friction, the frequency of these oscillations almost coincides with the *natural frequency* ω_0, which depends on the torsion spring constant D and the moment of inertia J of the flywheel: $\omega_0 = \sqrt{D/J}$.

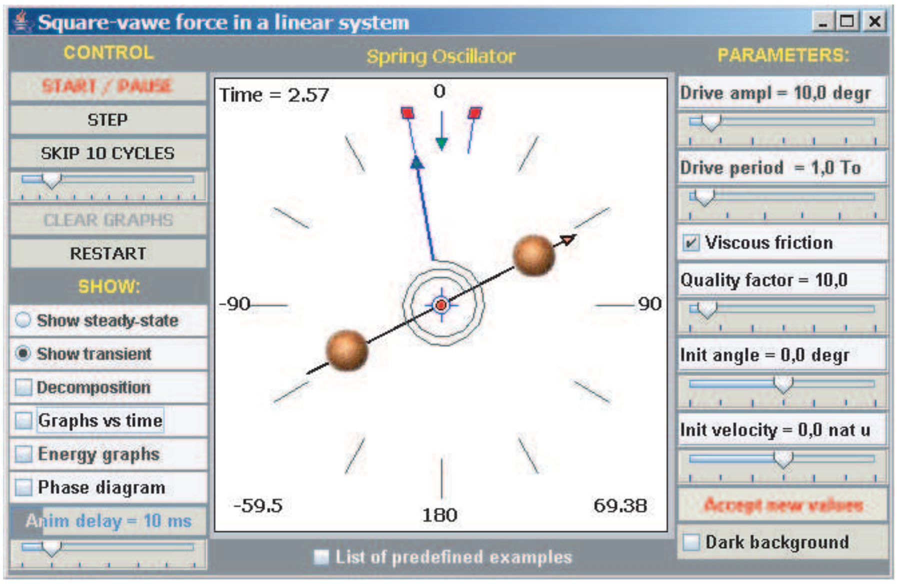

Figure 1: Schematic image of the torsion spring oscillator on the computer screen.

An external torque whose shape is that of a periodic square-wave can be realized by abruptly displacing the driving rod alternately in opposite directions through the same angle in equal time intervals. It is assumed that the displacements of the rod and thus of the equilibrium position of the flywheel occur so quickly that there is no significant change in either the angular displacement or velocity of the flywheel during the displacement of the rod. In the mathematical model of the system these forced displacements of the driving rod are set to occur instantaneously.

Any displacement of the equilibrium position of the flywheel is equivalent to the exertion of an additional constant external torque. Thus, these periodic abrupt movements of the rod in effect exert an external periodic piecewise constant driving torque on the flywheel. A torsion oscillator excited by such a square-wave torque provides a convenient example by which we can study a linear oscillatory system under the action of a periodic but *non-sinusoidal* external force.

1.2 The Differential Equation of Forced Oscillations

We let the instantaneous displacements of the rod occur alternately to the right and to the left after the lapse of equal time intervals $T/2$, so that during the interval $(0,\ T/2)$ the equilibrium position of the flywheel is displaced to the right through a fixed angle ϕ_0, and during the next interval $(T/2,\ T)$ the equilibrium position is displaced to the left through the same angle. Thus, T is the full period of the external non-sinusoidal action, repeated indefinitely.

When the system is at rest, we let the needle showing the position of the flywheel be parallel to the driving rod. In other words, the spring is not strained when the needle points in the same direction as does the rod. The zero point of the dial indicates the central position of the exciting rod (the vertical position on the screen). The angle of deflection φ of the needle from this zero point indicates the position of the flywheel. When the rod is deflected from the vertical position through an angle ϕ, and the flywheel at the same time is deflected through an angle φ, the spiral spring is twisted from its unstrained state through the angle $\varphi - \phi$. The spring then exerts a torque $-D(\varphi - \phi)$ on the flywheel which is proportional to this angle. Thus, the differential equation of rotation of the flywheel, whose moment of inertia about the axis of rotation is J, is given by:

$$J\ddot{\varphi} = -D(\varphi - \phi). \tag{1}$$

We transfer $-D\varphi$ (the part of the elastic torque which is proportional to φ) to the left side of Eq. (1), divide the resulting equation by J, and introduce the value $\omega_0 = \sqrt{D/J}$, whose physical meaning is the frequency of natural oscillations in the absence of friction. Thus we obtain:

$$\ddot{\varphi} + \omega_0^2\varphi = \omega_0^2\phi. \tag{2}$$

The right-hand side $\omega_0^2\phi$ of this equation can be treated as an external torque (divided by J) caused by the displacement of the rod from its central position through an angle ϕ. During the time interval $(0,\ T/2)$ the rod is displaced to the right so that $\phi = +\phi_0$. At the instant $T/2$ the rod abruptly turns to the left through the same angle ϕ_0 measured from the central position, and remains there during the subsequent interval $T/2$, so that for the second half of the driving period, $(T/2,\ T)$, the angle ϕ in Eq. (2) equals $-\phi_0$:

$$\ddot{\varphi} + \omega_0^2\varphi = \begin{cases} \omega_0^2\phi_0, & (0,\ T/2), \\ -\omega_0^2\phi_0, & (T/2,\ T). \end{cases} \tag{3}$$

In the presence of viscous friction, a term proportional to the angular velocity $\dot{\varphi}$ is added to the differential equation, Eq. (3), for forced oscillations:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \begin{cases} \omega_0^2\phi_0, & (0,\ T/2), \\ -\omega_0^2\phi_0, & (T/2,\ T). \end{cases} \tag{4}$$

Here the damping constant γ characterizes the strength of viscous friction in the system. Because of friction, natural oscillations of the flywheel gradually damp out, and a while after the external force began to act, a steady-state periodic motion of the flywheel is eventually established with a period equal to the period T of the driving force. The greater the decay time τ of natural oscillations ($\tau = 1/\gamma$), the longer the duration of this transient process.

We recall that in the case of a sinusoidal driving torque, the steady-state oscillations of the flywheel acquire not only the period of the external action but also the sinusoidal pattern of its functional time dependence. However, as we shall next see, a periodic driving force whose time dependence is something other than sinusoidal produces a steady-state response with the same period but with a time dependence different from that of the driving force.

1.3 Harmonics of the Square-wave Driving Force and of the Steady-state Response of the Oscillator

We consider below two different ways of determining the steady-state response of the oscillator to a non-harmonic driving force, such as the square-wave time-dependent force discussed above.

One way is based on the decomposition of the time dependence of the external force in a Fourier series, i.e., on the representation of this force as a superposition of sinusoidal components called *harmonics.* Because the differential equation of motion for the spring oscillator is linear, the influence of each of these harmonic components of the external force can be considered independently. Each sinusoidal component of the driving torque produces its own sinusoidal response of the same frequency in the motion of the flywheel. The amplitude and phase of each sinusoidal response can be calculated separately.

The net steady-state forced motion of the flywheel can be found as the superposition (i.e., the sum) of these individual responses. Thus, to each sinusoidal component of the periodic driving force (to the *input harmonic*), there corresponds a sinusoidal component of the same frequency in the steady-state motion of the responding oscillator (the *output harmonic*). Since the relative contributions of harmonic components to this response are different from the corresponding contributions to the driving force, the graph of the net motion of the flywheel has a different shape than does the graph of the motion of the driving rod.

In particular, it may occur that one of the input harmonics with a relatively small amplitude induces an especially large amplitude in the output oscillations. Such is the case when the frequency $n\omega$ of this harmonic is close to the natural frequency ω_0 of the oscillator since forced oscillations caused by this sinusoidal force occur under conditions of resonance. On the other hand, the relative contributions of the input harmonics whose frequencies lie far from the maximum of the resonance curve, are considerably attenuated in the output oscillations. In the output steady-state oscillations such harmonics are appreciably suppressed. The oscillator responds *selectively* to sinusoidal external forces of different frequencies.

Differences between the pattern of the time dependence of output steady oscillations and that of the input driving force are caused not only by changes from input to output in the relative amplitudes of the different harmonics but also by changes in their phases. If the quality factor Q of the responding oscillator is very large ($Q \gg 1$), its resonance curve (amplitude versus frequency) is very sharp, and the dependence of phase on frequency is nearly a step-function. Specifically, all harmonic components whose frequencies ω_k are lower than the resonant frequency ω_0 in the output oscillations of the flywheel are nearly in phase with the corresponding components of the input driving force. But harmonics whose frequencies ω_k are higher than the resonant frequency contribute to the output oscillations with nearly inverted phases: in the Fourier decomposition of the output steady-state oscillations, their phases are almost opposite to the phases of the corresponding harmonics of the driving force. If there is a sinusoidal component in the driving force whose frequency is near ω_0, this harmonic produces a significantly increased relative contribution to the output oscillations of the flywheel. This output harmonic lags in phase by $\pi/2$ behind the corresponding input harmonic in the spectrum of the driving force.

The analytic expression for Eq. (4) for which the square-wave shape of its right-hand side has been Fourier decomposed has the following form:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \sum_{k=1,\,3,\,5\ldots}^{\infty} \frac{4\phi_0\omega_0^2}{\pi k}\,\sin\omega_k t. \tag{5}$$

The Fourier series of the piecewise-constant external force in Eq. (4) contains only odd harmonics with frequencies $\omega_k = k\omega$ ($k = 1,\ 3,\ 5,\ \ldots$), where $\omega = 2\pi/T$, the frequency of the driving force. We note that the amplitudes of harmonics of the square-wave function decrease rather slowly, as $1/k$, with the increase of their index k and their frequency ω_k. This case is a very good example of a multi-harmonic external excitation of the oscillator since the frequency spectrum of the square-wave driving force is rich in harmonics.

The particular periodic solution of the linear differential equation, Eq. (5), gives the following time dependence for the angular displacement, $\varphi(t)$, for steady-state forced oscillations:

$$\varphi(t) = \sum_{k=1,\,3,\,5\ldots}^{\infty} \frac{4\phi_0}{\pi k}\frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega_k^2)^2+4\gamma^2\omega_k^2}}\,\sin(\omega_k t+\alpha_k). \tag{6}$$

where the phases α_k of the individual harmonics are determined by:

$$\tan\alpha_k = \frac{2\gamma\omega_k}{\omega_k^2-\omega_0^2}. \tag{7}$$

Equations (6) and (7) display clearly the peculiarities of the response of the oscillator to the square-wave driving action of the rod discussed above. In particular, when the frequency ω_k of one of the harmonics of the external force is near the resonant frequency ω_{res} of the oscillator, there is a resonant response from the oscillator because the denominator in the corresponding term of the sum in Eq. (6) is very small, especially for a weakly damped oscillator whose damping constant is small ($\gamma \ll \omega_0$), that is, whose quality factor Q is large ($Q = \omega_0/(2\gamma) \gg 1$).

For weak and moderate friction the resonant frequency is very near the natural frequency ω_0:

$$\omega_{\mathrm{res}} = \sqrt{\omega_0^2-2\gamma^2} = \omega_0\sqrt{1-\frac{2\gamma^2}{\omega_0^2}} \approx \omega_0\left(1-\frac{\gamma^2}{\omega_0^2}\right).$$

Since the fractional difference between ω_{res} and ω_0 is of the second order in the small parameter $\gamma/\omega_0 = 1/(2Q)$, in most cases of practical importance we need not distinguish the resonant frequency from the natural one. That is, we can assume that $\omega_{\mathrm{res}} = \omega_0$.

Figure 2 illustrates the transformation of the input spectrum of an external square-wave force into the output spectrum of the steady-state response of the oscillator. The resonance curve in the upper left corner shows how the oscillator with certain given parameters responds to the individual harmonic components of the external force.

When the frequency of the *sinusoidal* external force is varied slowly (a *sweep frequency*), the resonant response of the oscillator can occur at only one driving frequency ω, namely $\omega = \omega_{\mathrm{res}}$, the resonant frequency of the oscillator. In other words, in the case of sinusoidal excitation there is only one resonance, and it occurs when the driving period T equals the natural period T_0 of the oscillator.

However, in the case of the square-wave excitation, resonance occurs each time the driving period T is an odd-number multiple of the natural period T_0 of the oscillator, that is, when $T = (2n+1)T_0$, where $n = 0,\ 1,\ 2,\ \ldots$ Resonances, for which $n > 1$, occur when the frequency of one of the odd harmonics of the square-wave driving torque approaches the resonant frequency of the oscillator.

A linear oscillator with a large quality factor Q (i.e., with a sharp resonance curve) can appreciably respond only to the single harmonic component of a complex external force whose frequency is very near the resonant frequency of the oscillator. In this respect such an oscillator can be regarded as a

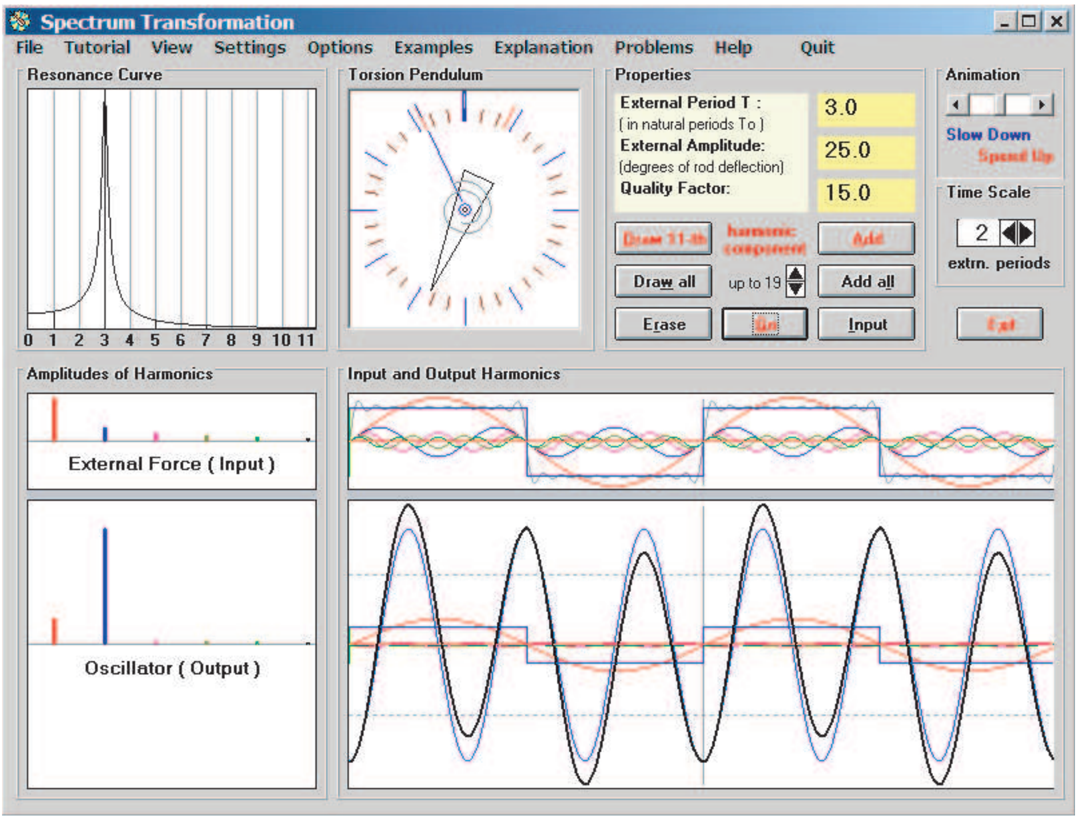

Figure 2: Transformation of the spectrum of the input square-wave external torque into the spectrum of steady-state output oscillations.

spectral instrument which selects a definite spectral component of an external action. That is, if we cause to "sweep" the natural frequency of oscillator through a range of frequencies, such oscillator responds resonantly to the complex external input each time its natural frequency coincides with one of the harmonic frequencies in the Fourier expansion of the external force. In other words, a sweep-frequency oscillator with a large quality factor provides us with a means by which a complex periodic input can be physically decomposed into its Fourier components.

The mathematical representation of the square-wave function on the right-hand side of Eq. (4) is not unique. The function can be represented as a sum of other functions in many different ways. That is, it is possible to express the external action either as a Fourier series of sine and cosine functions or as a series of other complete sets of functions. From the mathematical point of view, all such decompositions are equally valid. The usefulness of the Fourier decomposition is associated with *physics*. It arises from the capability of a linear harmonic oscillator to perform this decomposition physically.

1.4 Forced Oscillations as Natural Oscillations About the Alternating Equilibrium Positions

Another way to obtain an analytic solution to the differential equation of motion, Eq. (4), for steady-state oscillations forced by the square-wave external torque, is based on viewing the steady-state motion as a sequence of free oscillations which take place about an equilibrium position that periodically alternates between $+\phi_0$ and $-\phi_0$. During the first half-cycle, from $t = 0$ to $T/2$, the equilibrium position is located at $\varphi = +\phi_0$. For this half-cycle the general form of the dependence of $\varphi(t)$ on t can be written as:

$$\varphi(t) = \phi_0 + Ae^{-\gamma t}\cos(\omega_1 t + \theta), \qquad (0,\ T/2), \tag{8}$$

where $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$ is the frequency of damped natural oscillations, and A and θ are arbitrary constants of integration determined by conditions at the beginning of the half-cycle. During the next half-cycle $(T/2,\ T)$, damped natural oscillations occur about an equilibrium position displaced through the same angle ϕ_0 but in the opposite direction. For this half-cycle the time dependence of $\varphi(t)$ has the form:

$$\varphi(t) = -\phi_0 - Ae^{-\gamma(t-T/2)}\cos(\omega_1(t - T/2) + \theta), \qquad (T/2,\ T), \tag{9}$$

where the constants A and θ have the same values as they do in Eq. (8). These values follow from the fact that, in steady-state oscillations, the graph of time dependence during the second half-cycle (when the driving rod is displaced to the left) must be the mirror image of the graph for the first half-cycle, shifted by $T/2$ along the time axis. This relationship can be clearly seen in Fig. 3, which shows the graphs of the time dependence of the angular deflection and the angular velocity in steady-state forced oscillations for $T = 3T_0$ (that is, when the period T of the driving force equals three natural periods T_0).

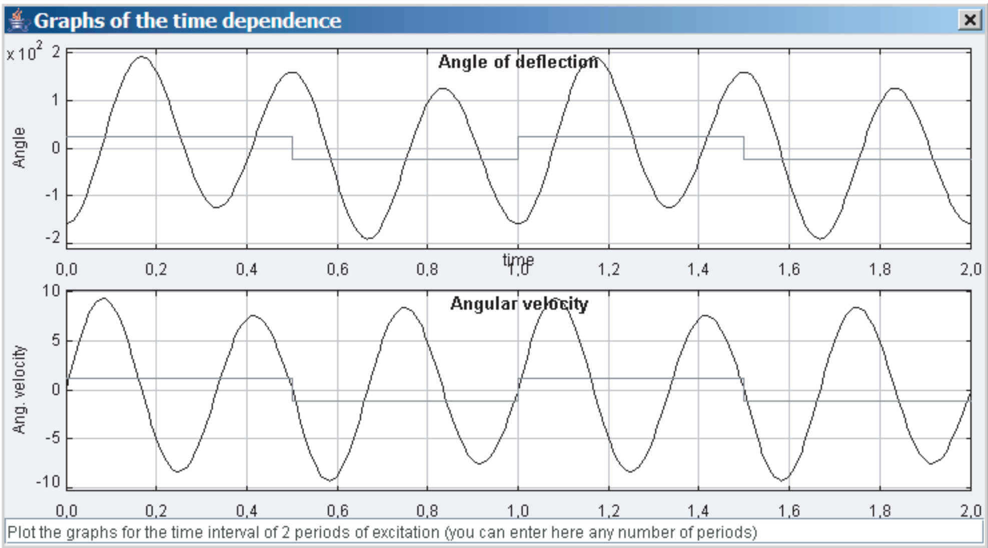

Figure 3: Plots of the time dependence of the deflection angle and the angular velocity for resonant steady-state oscillations at $T = 3T_0$.

The constants A and θ for any given values of T and γ can be calculated from the condition that during the instantaneous change in the positions of the driving rod, from one equilibrium position to the other, the angular deflection and the angular velocity of the flywheel do not change. This condition gives two simultaneous equations for A and θ. Solving the equations we find:

$$\tan\theta = -\frac{e^{-\gamma T/2}[\omega_1 \sin(\omega_1 T/2) + \gamma\cos(\omega_1 T/2)] + \gamma}{e^{-\gamma T/2}[\omega_1 \cos(\omega_1 T/2) - \gamma\sin(\omega_1 T/2)] + \omega_1} \tag{10}$$

and

$$A = \frac{2\phi_0}{e^{-\gamma T/2}\cos(\omega_1 T/2 + \theta) + \cos\theta}. \tag{11}$$

Equations (8)–(11) describe the steady-state motion only during the time interval from 0 to T. That is, if we substitute a value of t greater than T into these equations, they do not give the correct value for

$\varphi(t)$. Nevertheless, we can find the value of $\varphi(t)$ for an arbitrary t by taking into account that $\varphi(t)$ is a periodic function of t: $\varphi(t+T) = \varphi(t)$. Thus, having obtained the graph of $\varphi(t)$ for the time interval $[0,\ T]$, we can simply translate the graph to the adjacent time intervals $[T,\ 2T],\ [2T,\ 3T]$, and so on.

1.5 Transient Processes under the Square-wave External Torque

The above treatment of considering forced oscillations that are excited by a square-wave external torque as natural oscillations about alternating equilibrium positions, enables us to understand characteristics of both steady-state oscillations and transient processes. In particular, this treatment enables us to understand the physical reason for the resonant growth of the amplitude when the period of the driving force equals the natural period of the oscillator or when it equals some odd-number multiple of that period.

Suppose that before the external square-wave torque is applied, the oscillator has been at rest in its equilibrium position, $\varphi = 0$. When, at $t = 0$, the driving rod abruptly turns into a new position, ϕ_0, the flywheel, initially at rest, begins to execute damped natural oscillations about the new equilibrium position at ϕ_0 with the frequency $\omega_1 \approx \omega_0$. This oscillation begins with the an initial velocity of zero. As long as the rod remains at ϕ_0, the time dependence of the angular displacement of the flywheel, $\varphi(t)$, is

$$\varphi(t) = \phi_0 - \phi_0 \exp(-\gamma t) \cos \omega_1 t.$$

(For weak friction, $\omega_1 \approx \omega_0$). That is, the flywheel, starting out with $\varphi = 0$ at $t = 0$, passes through the equilibrium position $\varphi = \phi_0$ when $\omega_0 t = \pi/2$, and reaches its extreme deflection of nearly $\varphi = 2\phi_0$ at $\omega_0 t = \pi$. (Damping prevents it from quite reaching $\varphi = 2\phi_0$.) If the period T of the driving rod equals T_0, the flywheel arrives at the extreme point $\varphi \approx 2\phi_0$ and its angular velocity becomes zero just at the moment $t = T/2$, when one half of the driving period has elapsed. At this moment the rod instantly moves to the new position $-\phi_0$, which becomes the new equilibrium position of the flywheel for the next time interval ($T_0/2,\ T$). Hence, the next half-cycle of its natural oscillation starts again with an angular velocity of zero, but its initial angular displacement from the new central point is nearly $3\phi_0$. This value is nearly $2\phi_0$ greater than its value in the preceding half-cycle. (It would be exactly $2\phi_0$ greater in the absence of friction.) Thus, in the absence of friction, the amplitude of oscillation increases by the same value $4\phi_0$ during each full cycle of the external force, provided the driving period equals the natural period of the oscillator (or some odd-number multiple of that period).

In a real system such an unlimited growth of the amplitude linearly with time is impossible because of friction. The growth of the amplitude is approximately linear during the initial stage of the transient process. This resonant growth gradually decreases, and steady-state oscillation are eventually established, during which the increment of the amplitude occurring at every instantaneous displacement of the equilibrium position (at any jump of the driving rod) is nullified by an equal decrement caused by viscous friction during the intervals between successive jumps.

Such a process of gradual growth of the amplitude, which eventually results in oscillations of constant amplitude, is depicted very clearly by the phase trajectory shown in Fig. 4. If the oscillator is at rest in the equilibrium position at the moment the external force is activated, the phase trajectory originates at the origin of the phase plane. Its first section is a portion of a spiral that winds around a focus located at the point ($+\phi_0,\ 0$). This focus corresponds to the new equilibrium position displaced to the right. The next section, a continuation of the phase trajectory, describes damped natural oscillation about the other equilibrium position after the driving rod has jumped to the left. This is a segment of a similar spiral which winds around the symmetrical point ($-\phi_0,\ 0$) of the phase plane.

If the period of the square-wave external action equals an odd-numbered multiple of the natural period, the transition of the representative point from one spiral, centered say at ($+\phi_0,\ 0$), to the adjoining spiral centered at the other focus ($-\phi_0,\ 0$), occurs at a point of the φ-axis to the right of ϕ_0, at a maximal

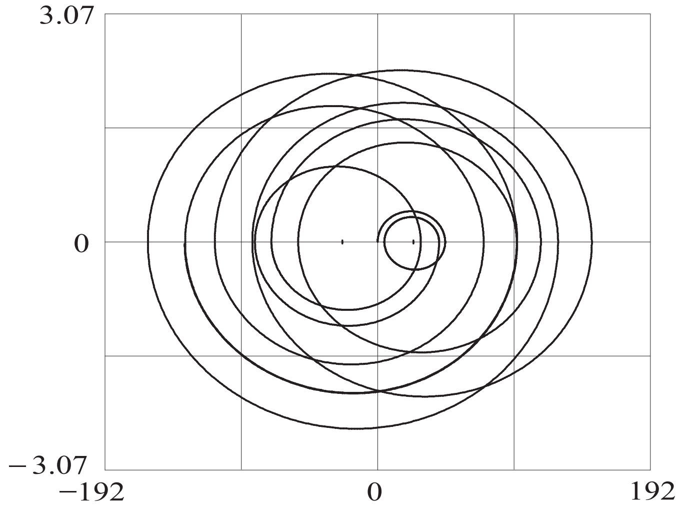

Figure 4: The phase diagram for the transient process of excitation from equilibrium for resonance occurring at $T = 3T_0$.

distance from the new focus. As a result, the new loop of the phase trajectory turns out to be larger than the preceding one. Such untwisting of the phase trajectory continues at a decreasing rate until the expansion of loops due to the alternation of the foci is nullified by their contraction caused by viscous friction. Eventually a closed phase trajectory is formed which corresponds to steady-state oscillations. This curve has a central symmetry about the origin of the phase plane. It consists of two branches, each representing damped natural oscillations about one of the two alternating symmetrical equilibrium positions.

1.6 Estimation of the Amplitude of Steady Oscillations

Using the arguments suggested in the previous section, we next evaluate the maximal angular deflection, φ_m, attained in the steady-state oscillations when the oscillator is driven by a square-wave external torque whose period T is an integral multiple of the natural period T_0 of the oscillator.

We consider first the main resonance in which the driving period equals the natural period: $T = T_0$. The estimation of φ_m can be made in the following way. The closed phase trajectory for the steady-state oscillation consists in this case of a single loop intersecting the φ-axis at the points $-\varphi_m$ and φ_m which are the points of extreme angular displacements of the oscillator. The angular separation of these points from the equilibrium position at ϕ_0 are $\varphi_m + \phi_0$ (on the left side of ϕ_0) and $\varphi_m - \phi_0$ (on the right side of ϕ_0).

The upper part of the phase trajectory is a half-loop of a spiral whose focus is at the point $+\phi_0$, displaced to the right from the origin. While the representative point passes along this upper half-loop from $-\varphi_m$ to φ_m, the oscillator executes one half of a period of damped natural oscillation about the equilibrium position ϕ_0, displaced to the right side: the flywheel passes from the extreme deflection $|\varphi_m + \phi_0|$ on the left side (measured from ϕ_0) to the extreme deflection $\varphi_m - \phi_0$ on the right side. When the oscillator reaches this extreme point, the equilibrium position switches to the focus $-\phi_0$, and the representative point then passes along the lower half-loop, thus closing the phase trajectory of the steady-state motion.

The relative decrease of the amplitude because of viscous friction during one half of the natural period ($t = T_0/2$) equals $\exp(-\gamma T_0/2)$. So the left and right separations for the upper half-loop are related to one another through this exponential factor giving the frictional decay for a half-cycle:

$$(\varphi_m + \phi_0)\exp(-\gamma T_0/2) = \varphi_m - \phi_0. \tag{12}$$

In the case of weak friction $\exp(-\gamma T_0/2) \approx 1 - \gamma T_0/2$. Using this approximation in Eq. (12) and solving for φ_m, we obtain the desired estimate:

$$\varphi_m \approx \phi_0 \frac{2}{\gamma T_0/2} = \phi_0 \, \frac{4}{\pi} Q. \tag{13}$$

The product of the damping constant γ and the natural period T_0 is expressed here in terms of the quality factor Q: $\gamma T_0/2 = \gamma\pi/\omega_0 = \pi/(2Q)$, since $Q = \omega_0/(2\gamma)$.

Equation (13) shows that for resonance induced by the fundamental harmonic of the square-wave external torque ($T = T_0$) the amplitude of steady-state oscillation is Q times greater than the amplitude $(4/\pi)\phi_0$ of this harmonic component in the square-wave motion of the rod. (See Eq. (5).) The same conclusion can be reached from a spectral approach to the treatment of stationary forced oscillations.

Through a similar calculation we can obtain an estimate of the maximal displacement, φ_m, attained in steady-state oscillations for any of the higher resonances when the period of the square-wave external torque is an odd multiple of the natural period. (See Problem 1.4.)

When the driving period is an even-numbered multiple of the natural period, the maximal deflection of the flywheel φ_m attained in steady-state forced oscillations can not exceed $2\phi_0$. We can easily see this result from the shape of the corresponding phase trajectory: each of its two symmetrical halves consists of an integral number of shrinking loops of a spiral winding around one of the foci ϕ_0 and $-\phi_0$. In figure 5 such a phase trajectory corresponds to $T = 2T_0$.

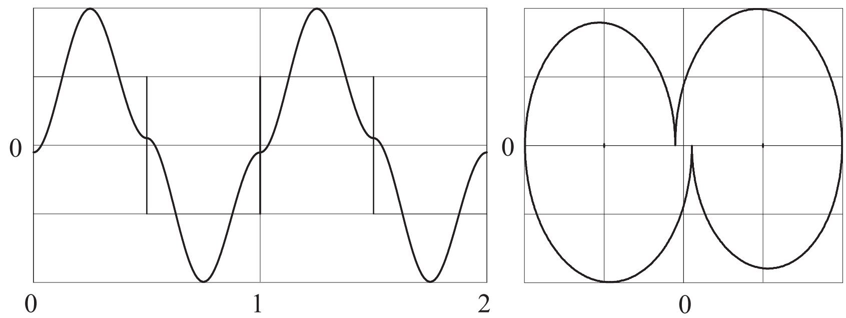

Figure 5: The plots of the time dependence of the angular deflection and the phase diagram for steady-state forced oscillations at $T = 2T_0$

For $T = 2T_0$, exactly one cycle of natural oscillation occurs while the equilibrium position is displaced to one side. In the absence of friction both closed loops of the steady-state phase trajectory meet at the origin of the phase plane, and the magnitude φ_m of the maximal displacement on each side of the zero point equals $2\phi_0$. Friction causes the loops to shrink, and the maximal displacement φ_m becomes less than $2\phi_0$ (Fig .5).

1.7 Steady Oscillations during Short-Period Displacements of the Driving Rod

For high frequencies of the external force, when the square-wave period T of the driving rod is very short compared to the natural period T_0 of the oscillator, in steady-state motion the flywheel executes only small vibrations about the mid-point $\varphi = 0$. The period of these vibrations is the same as that of the driving rod. However, they are 180° out of phase with the motion of the rod, and their amplitude is small compared to the amplitude ϕ_0 of the rod.

Since the flywheel moves little while the position of the rod is fixed at either ϕ_0 or $-\phi_0$, we can consider the torque of the spring exerted on the flywheel as nearly constant during the intervals between jumps of the rod. Indeed, there is little change in the deformation of the spring during the short interval between jumps. This elastic torque of the spring reverses direction at each jump of the rod, but its magnitude after a jump is almost the same as before the jump.

For short-period jumps of the driving rod, the graph of the angular velocity in steady-state oscillations consists of nearly rectilinear segments. They correspond to the rotation of the flywheel with a uniform angular acceleration caused by the constant torque of the strained spring. Such rotation continues in one direction during time intervals between jumps of the rod. After each succeeding jump the acceleration changes sign, remaining nearly the same in magnitude. In the graph of the angular velocity the straight segments join to form a saw-toothed pattern of isosceles triangles. The corresponding graph of the angular deflection is formed by alternating adjoining parabolic segments (see figure 6).

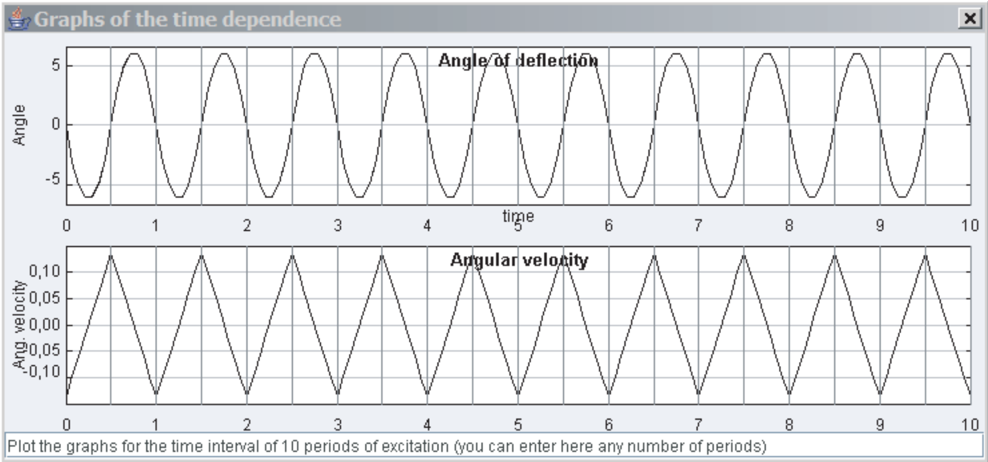

Figure 6: Steady-state oscillations at $T < T_0$ (a sawtooth graph of the angular velocity and parabolic segments of the graph of the angular deflection).

1.8 Energy Transformations

While the driving rod is at rest, the external source does neither positive nor negative work on the oscillator. Consequently, the exchange of mechanical energy between the oscillator and the source of the external driving force occurs only at the instants when the driving rod jumps from one position to the other. During the intervals between such jumps, while the oscillator executes damped natural oscillations about one of the two displaced equilibrium positions, there is no exchange of energy with the exciter, and only an alternating partial conversion between the elastic potential energy of the strained spring and the kinetic energy of the flywheel occurs, accompanied by the gradual dissipation of mechanical energy because of friction. Time-dependent graphs of the energy transformations at steady-state oscillations with $T = 3T_0$ are shown in figure 7.

It is clear that in this model of the physical system (in which the jumps of the rod are set to be instantaneous) the angular velocity of the massive flywheel does not change during a jump. Therefore the kinetic energy of the flywheel also does not change. An abrupt change occurs only in the value of the elastic potential energy of the spring. This potential energy increases if after a jump of the rod the spring tension increases, and the potential energy decreases if the spring tension decreases.

To get a general idea of these energy transformations, you can open the window "Phase diagram" in

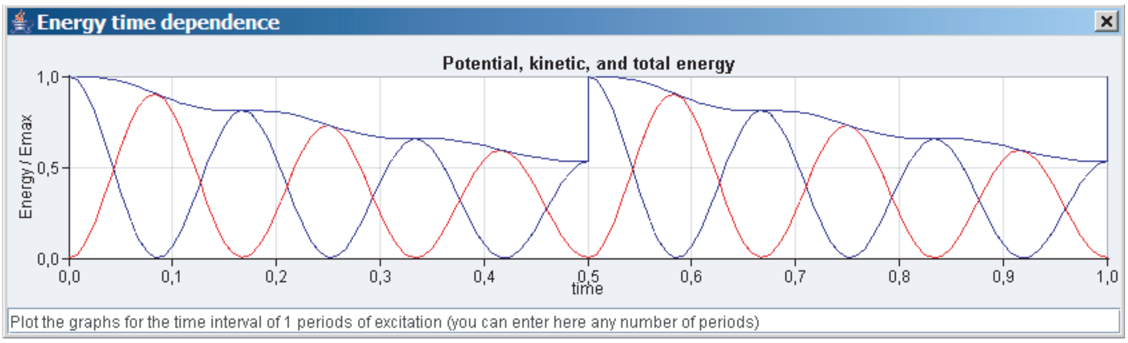

Figure 7: Energy transformations at steady-state oscillations with $T = 3T_0$

the simulation program and there consider the motion of the point representing the total energy in the graph of potential energy versus the angle of deflection.

A parabolic potential well corresponds to each of the two equilibrium positions of the oscillator. When the flywheel is located at the angle φ from its central position, the corresponding potential energy of the spring is given by one of the two quadratic functions:

$$U(\varphi) = \frac{1}{2}k(\varphi \mp \phi_0)^2. \tag{14}$$

We must take the upper sign in Eq. (14) if the equilibrium position is displaced to the right (to the point $+\phi_0$), and the lower sign if otherwise.

An instantaneous jump of the driving rod from one position to the other at a fixed value of the angle φ causes the representative point to make an abrupt vertical transition from one of these parabolic potential wells to the other. During the interval before the next jump, while the oscillator executes damped natural oscillations about a displaced equilibrium position, the point which represents the total energy travels back and forth between the walls of the corresponding potential well, descending gradually toward the bottom because of energy losses caused by friction. This behavior is clearly seen in Fig. 5.7.

It is important to note that in this simplified model of the physical system the deformation of the spiral spring is assumed to be quasistatic. In other words, we ignore the possibility that the spring vibrates as a distributed parameter system, each part having both elastic and inertial properties. For a light spring (acting on a coparatively massive flywheel) these vibrations are characterized by much higher frequencies than the frequencies of the torsional oscillations of the flywheel. These rapid vibrations of the spring quickly damp out.

1.9 The Electromagnetic Analogue of the Mechanical System

Forced oscillations of the electric charge q stored in a capacitor in a resonant series LCR-circuit excited by a square-wave input voltage $V(t)$ obey the same differential equation as does the forced oscillation of a mechanical torsion spring oscillator excited by periodic abrupt changes of position of the driving rod:

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = \omega_0^2 CV(t). \tag{15}$$

In this equation ω_0 is the natural frequency of oscillations of charge in the circuit in the absence of resistance. It depends on the capacitance C of the capacitor and the inductance L of the coil: $\omega_0 = 1/\sqrt{LC}$. The damping constant $\gamma = R/(2L)$ characterizes the dissipation of electromagnetic energy occuring in a resistor whose resistance is R.

Because of this similarity, the mechanical system simulated in the computer program enables us to give a very clear treatment of transformation of the square-wave input voltage $V(t) = \pm V_0$ into the output voltage $V_C(t) = q/C$ (voltage across the capacitor C). The output voltage $V_C(t)$ is analogous to the deflection angle $\varphi(t)$ of the rotor. The alternating electric current $I(t) = \dot{q}(t)$ in the circuit is analogous to the angular velocity $\dot{\varphi}(t)$ of the mechanical model.

The assumption concerning quasistatic character of the spring deformation in the mechanical model, i.e., the assumption of possibility to neglect rapid vibrations of the spring as a distributed parameter system, corresponds to the ordinary implicit assumption of quasistationary current in the circuit. According to this assumption, the momentary value of the current is the same along the whole circuit. The assumption is valid if the inductance and the capacitance of the wires are negligible compared with the inductance of the coil and the capacitance of the capacitor respectively. In this case the oscillatory circuit can be treated as a system with lumped parameters (as a system with one degree of freedom), in which all the capacitance is concentrated in the capacitor and all the inductance is concentrated in the coil.

Some caution is necessary in interpreting the analogy between the mechanical oscillator and the electric LCR-circuit with respect to energy transformations. It is incorrect to identify exactly the electric potential energy of a charged capacitor with the elastic potential energy of a strained spring because the latter depends directly on the angle $(\varphi \pm \phi_0)$, while the energy of a capacitor depends directly on the charge q or on the corresponding voltage $V_C = q/C$ (*not* on the voltage $(V_C \pm V_0)$). In contrast to the spring oscillator, for which a jump in the position of the rod causes an abrupt change of the elastic potential energy of the spring, a jump in the input voltage across an electric circuit does not abruptly change the charge and the energy of a capacitor.

The spectral treatment of the transformation of a square-wave driving force (an input) into the steady-state oscillations of the mechanical spring oscillator (the output) is equally valid for the transformation of an input square-wave voltage into the output oscillations of charge in the analogous electric circuit. Only those harmonics of the input signal which are near the resonant frequency of the circuit are noticeably present in the output voltage across the capacitor.

A resonant circuit selectively transmits to the output signal those harmonic components of the input signal whose frequencies are near the resonant frequency of the circuit. The greater the quality factor Q, the sharper the resonance curve and the finer the *selectivity* of the oscillatory circuit.

It is possible to vary the natural frequency $\omega_0 = 1/\sqrt{LC}$ of a resonant circuit by varying either the capacitance C or the inductance L. Such tunable resonant circuits with high selectivity can serve as spectral instruments that are physically able to Fourier analyze a complex input signal.

2 Questions, Problems, Suggestions

2.1 Swinging of Oscillator at Resonance

1.1 **The Principal Resonance in the Absence of Friction.** Select the idealized case of no friction. Enter the period T of the external force, letting this period be the period T_0 of natural oscillations. Choose null initial conditions. That is, let the oscillator be at rest in the equilibrium position at the moment the external force is activated.

(a) What must be the value ϕ_0 of the angular amplitude of the square-wave motion of the driving rod in order that the amplitude reach 180° after the first 10 cycles? Verify your answer by a simulation experiment.

(b) What regularity does the growth of the amplitude exhibit? Explain the form of the phase trajectory displayed. How does the energy of oscillator grow with time?

(c) If the oscillator is exactly tuned to resonance, is it possible for the amplitude to diminish? Give some physical justification for your answer. Can you test your answer by a simulation experiment?

1.2 **High Resonances in the Absence of Friction.** Examine the resonant excitation of the oscillator initially at rest in the equilibrium position when the period of the external square-wave force is three times longer than the natural period: $T = 3T_0$.

(a) At what value ϕ_0 of the amplitude of a square-wave oscillation of the driving rod does the oscillator reach its maximal deflection of 180° in 10 first cycles of the external action? Verify your prediction experimentally.

(b) What are the differences in the graphs and the phase trajectories between this case and the previous case (Problem 1.1) for which $T = T_0$?

1.3* **Transient Process and Steady-state Oscillations at the Principal Resonance.** Letting the period of the square-wave motion of the driving rod be at the principal resonance, $T = T_0$, and letting the flywheel be initially at rest in the equilibrium position, examine the transient process and the steady-state oscillations in the presence of friction:

(a) Calculate the amplitude of steady-state oscillations for the values $\phi_0 = 10^\circ$ and $Q = 10$. Verify your result experimentally.

(b) What regularity does the growth of the amplitude exhibit in this case? Explain the peculiarities of the phase trajectory.

(c) What is the initial amplitude of damped natural oscillations constituting the transient process for $\phi_0 = 10^\circ$ and $Q = 10$?

(d) What initial conditions cause steady-state forced oscillations to appear from the moment the external force begins to act, thus eliminating the transient process? Verify your answer experimentally.

(e) Examine the spectrum of steady-state oscillations (the output) in this case. Why are these oscillations nearly purely harmonic in spite of the non-sinusoidal, square-wave shape of the input?

1.4* **Steady-state Oscillations at High Resonances.**

(a) Calculate the amplitude of steady-state oscillations for $\phi_0 = 25^\circ$, $Q = 5$, and $T = 3T_0$. Explain the shape of the graphs displayed and of the phase trajectory.

(b) What energy transformations take place during steady-state oscillations? Compare the graphs of the time dependence of the kinetic, potential, and total energy with the corresponding graphs of the angular deflection and the angular velocity of the flywheel. Explain the shape of the graph of the total energy versus the angle of deflection, and explain its relationship to the parabolic potential wells shown in the same diagram.

(c) Which harmonic components determine the shape of output steady-state oscillations in this case? Why, in spite of the exact tuning of the oscillator to the frequency of the third harmonic of the input external force, does the first harmonic component of this force appreciably influence the shape of the output oscillations? How does this harmonic exhibit itself in the pattern of the output oscillations?

(d) Examine the influence of friction on the shape and on the spectral composition of steady-state oscillations at $T = 3T_0$. Note the relative reduction in the contribution of the first and fifth harmonics as the quality factor of the oscillator is increased.

(e) Explore the resonant oscillations of the flywheel when the frequency of the fifth or the seventh harmonic of the external square-wave driving force coincides with the natural frequency of the oscillator. Observe the transformation of the spectrum from input to output, and the dependence of the spectrum of steady-state oscillations on the quality factor of the oscillator. What is the shape of the phase trajectory in these cases? How can you estimate the value of the maximal displacement of the flywheel from the mid-point of its oscillations (from the zero point of the dial) when friction is large (when Q ranges say from 1 to 3)?

2.2 Nonresonant Forced Oscillations

2.1* **Conditions which Eliminate a Transient at** $T = 2T_0$.

(a) Predict the shape of the graphs of the angular deflection, angular velocity, and the phase trajectory of forced oscillations in the absence friction, for $T = 2T_0$. Under what initial conditions is the steady-state oscillation established immediately after the force is activated? Verify your predictions in a simulation experiment.

(b) Why does the total energy remain constant in these oscillations? When the driving rod executes a jump, why does the oscillator neither gain energy from nor give back energy to the external source?

(c) Examine the spectral composition of steady-state oscillations. Note the contribution of the third harmonic and the influence of its alteration in phase on the shape of the output oscillations: The frequency of the third harmonic in this case is higher than the natural frequency of the oscillator. Therefore its phase in the output oscillations is inverted. As a result, this harmonic component, superimposed on the fundamental, produces a time- dependent graph with bulges at the positions of the flat parts of the square-wave input graph.

2.2 **Steady-state Oscillations for** $T = 2T_0$.

(a) Consider forced oscillations for the case in which $T = 2T_0$ in the presence of friction, by setting $Q \approx 5 - 10$. How do the time-dependent graphs and the phase trajectory differ from the preceding case (Problem 2.1), in which friction is absent? Calculate the maximal deflection of the flywheel attained in these steady-state oscillations. Note changes in the energy transformations.

(b) Why does friction not noticeably influence the spectral composition of steady-state oscillations in this case, in contrast to the case in which $T = 3T_0$?

2.3 **Steady-state Oscillations for a Large External Period.**

(a) Examine forced oscillations for a case in which the natural frequency of the oscillator lies somewhere between the frequencies of two consecutive high odd harmonics of the external action (e.g., let $5T_0 < T < 7T_0$). Which harmonic components dominate in the output steady-state oscillations? Compare the shape of the output steady-state oscillations with the shape of the input square-wave impulses. What is the main difference between the patterns of input and output oscillations?

(b) Investigate the influence of friction on the character of steady-state oscillations. Why are the distortions of the output less prominent the stronger the damping? That is, why is the shape of the output curve for large friction nearly rectangular?

(c) Explain the energy transformations in these oscillations using the graph of the total energy versus the angle of deflection. What is the relationship of this graph with the parabolic potential wells shown in the same diagram?

2.4 **Oscillations Forced by Short-Period Impulses.**

(a) Choose a value for the period T of the square-wave external force to be a small fraction (say $0.2 - 0.3$) of the natural period T_0 of the oscillator. The graph of the angular velocity versus time for steady-state output oscillations has a saw-toothed pattern with teeth which are nearly rectilinear isosceles triangles. Suggest an explanation. What is the difference between the graph of the angular deflection versus time for this case and a sine curve?

(b) Evaluate theoretically the height of a tooth of the angular velocity graph. Evaluate also the maximal deflection angle for these non-sinusoidal steady-state oscillations. Consider the case of weak or moderate friction. Let, for example, the period T be $T_0/4$ and the angle ϕ_0 describing the instantaneous deflections of the driving rod be $30°$. What spectral composition is characteristic of such oscillations?

2.3 Supplement: Review of the Principal Formulas

The differential equation of motion for a torsion spring oscillator driven by a square-wave external force:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \begin{cases} \omega_0^2\phi_0, & (0,\ T/2), \\ -\omega_0^2\phi_0, & (T/2,\ T). \end{cases}$$

The same equation in which the square-wave shaped right-hand side is represented as a Fourier series:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\varphi = \sum_{k=1,\,3,\,5\ldots}^{\infty} \frac{4\phi_0\omega_0^2}{\pi k}\ \sin\omega_k t.$$

The particular periodic solution of the equation (describing steady-state oscillations):

$$\varphi(t) = \sum_{k=1,\,3,\,5\ldots}^{\infty} \frac{4\phi_0}{\pi k} \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega_k^2)^2 + 4\gamma^2\omega_k^2}}\ \sin(\omega_k t + \alpha_k),$$

where the phases α_k of the individual harmonics are determined by:

$$\tan\alpha_k = \frac{2\gamma\omega_k}{\omega_k^2 - \omega_0^2}.$$

The time dependence of $\varphi(t)$ during the interval $0 \le t \le T/2$, when the equilibrium position is located at $\varphi = \phi_0$:

$$\varphi(t) = \phi_0 + Ae^{-\gamma t}\cos(\omega_1 t + \theta), \qquad (0,\ T/2),$$

where $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$, the frequency of natural damped oscillations, and A and θ are some constants.

The time dependence of $\varphi(t)$ during the interval $T/2 \le t \le T$, when damped natural oscillations occur about the equilibrium position located at $-\phi_0$:

$$\varphi(t) = -\phi_0 - Ae^{-\gamma(t-T/2)}\cos(\omega_1(t - T/2) + \theta), \qquad (T/2,\ T).$$

For a steady-state process, the constants A and θ here have the same values as they do for the interval $(0,\ T/2)$.