Parametric Excitation of a Linear Oscillator

Manual

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Annotation. The manual includes a description of the simulated physical system and a summary of the relevant theoretical material for students as a prerequisite for the virtual lab "Parametric Oscillations in a Linear System." The manual includes also a set of theoretical and experimental problems to be solved by students on their own, as well as various assignments which the instructor can offer students for possible individual mini-research projects.

Contents

1 Summary of the Theory

This manual is concerned with the phenomenon of *parametric resonance* arising from a periodic square-wave modulation of the moment of inertia of a linear torsion spring oscillator. Computer simulations of oscillations are accompanied by plotting the time dependencies of the angle of deflection and the angular velocity. Phase trajectories and energy transformations for different values of the degree of modulation are analyzed, and conditions of parametric regeneration and of parametric resonance are discussed. Ranges of frequencies where parametric excitation is possible are determined. Stationary oscillations on the boundaries of these ranges are investigated.

1.1 Classification of Oscillations

In the conventional classification of oscillations by their method of excitation, oscillations are called *forced* if an oscillator is subjected to an external periodic influence whose effect on the system can be expressed by a separate term, a periodic function of the time, in the differential equation of motion describing the system. Forced oscillations are discussed in Chapter 4 and Chapter 5 of this Manual.

The investigation of *nonstationary, position-dependent forces*, i.e., those which are explicitly determined by both temporal and spatial coordinates, is more complicated. For example, let a restoring force $F = -kx$ arise when the system is displaced through some distance x from the equilibrium position. But in contrast to the stationary case, the parameter k changes with time because of some periodic influence: $k = k(t)$. In the differential equation of motion for the system,

$$m\ddot{x} = -k(t)x, \tag{1}$$

the coefficient of x is not constant: it explicitly depends on time. Oscillations in such a system are essentially different from both free oscilations, which occur when k is constant, and forced oscillations, which occur when k is constant and an additional time-dependent forcing term is added to the right side of the equation of motion, Eq. (1).

In the case of *periodic* changes of the parameter k, when $k(t + T) = k(t)$, where T is the period, the corresponding differential equation, Eq. (1), is called *Hill's equation.* Oscillations in a system described by Hill's equation are called *parametrically excited* or simply *parametric.* When the amplitude of oscillation caused by the periodic modulation of some parameter increases steadily, we describe the phenomenon as *parametric resonance.* In parametric resonance, equilibrium becomes unstable and the system performs oscillations whose amplitude increases exponentially.

The characteristics and causes of parametric resonance are considerably different from those of the resonance occurring when the oscillator responds to a periodic external force. Specifically, the resonant relationship between the frequency of modulation of the parameter and the mean natural frequency of oscillation of the system is different from the relationship between the driving frequency and the natural frequency for the usual resonance in forced oscillations. And if there is friction, the amplitude of modulation of the parameter must exceed a certain threshold value in order to cause parametric resonance.

1.2 The Simulated Physical System

The suggested computer program simulates a simple physical system that perfectly suits to the initial acquaintance with the basics of parametric resonance, namely, a torsion spring oscillator (figure 1) similar to the balance device of a mechanical watch. It consists of a rigid rod which can rotate about an axis that passes through its center. Two identical weights are balanced on the rod. An elastic spiral spring is attached to the rod. The other end of the spring is fixed. When the rod is turned about its axis, the spring flexes. The restoring torque $-D\varphi$ of the spring is proportional to the angular displacement φ of the rotor from the equilibrium position. After a disturbance, the rotor executes natural harmonic torsional oscillations.

To provide modulation of a system parameter, we assume that the weights can be shifted simultaneously along the rod in opposite directions into other symmetrical positions so that the rotor as a whole remains balanced, but its moment of inertia J is changed. Periodic modulation of the moment of inertia by such mass redistribution can cause, under certain conditions, a growth of (initially small) natural rotary oscillations.

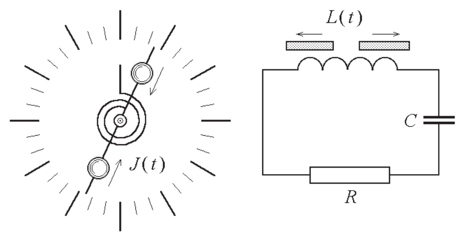


Figure 1: Schematic image of the torsion spring oscillator with a rotor whose moment of inertia is forced to vary periodically (left), and an analogous LCR-circuit with a coil whose inductance is modulated by moving periodically an iron core in and out of the coil.(right).

In the case of the square-wave modulation, abrupt, almost instantaneous increments and decrements of the moment of inertia occur sequentially, separated by equal time intervals. We denote these intervals by $T/2$, so that T equals the period of the variation in the moment of inertia (the *period of modulation*).

The square-wave variation of a parameter can produce considerable oscillation of the rotor if the period of modulation is chosen properly. For example, suppose that the weights are drawn closer to each other at the instant at which the rotor passes through the equilibrium position, when its angular velocity is almost maximal. While the weights are being moved, the angular momentum of the system remains constant since no torque is needed to effect this displacement. Thus the resulting reduction in the moment of inertia is accompanied by an increment in the angular velocity, and the rotor acquires additional energy. The greater the angular velocity, the greater the increment in energy. This additional energy is supplied by the source that moves the weights along the rod.

On the other hand, if the weights are instantly moved apart along the rotating rod, the angular velocity and the energy of the rotor diminish. The decrease in energy is transmitted back to the source.

In order that increments in energy occur regularly and exceed the amounts of energy returned, i.e., in order that, as a whole, the modulation of the moment of inertia regularly feed the oscillator with energy, the period of modulation must satisfy certain conditions.

For instance, let the weights be drawn closer to and moved apart from each other twice during one mean period of the natural oscillation. Furthermore, let the weights be drawn closer at the instant of maximal angular velocity. Then they are moved apart almost at the instant of extreme deflection, when the angular velocity is nearly zero. The angular velocity increases at the moment the weights come together, and vice versa. But if the angular momentum is zero at the moment the weights move apart, this particular motion causes no change in the angular velocity or kinetic energy of the rotor. Thus modulating the moment of inertia at a frequency twice the mean natural frequency of the oscillator generates the greatest growth of the amplitude, provided that the phase of the modulation is chosen in the way described above.

It is evident that the energy of the oscillator is increased most greatly not only when two full cycles of variation in the parameter occur during one natural period of oscillation, but also when two cycles occur during three, five or any odd number of natural periods.

We shall see later that the delivery of energy, though less efficient, is also possible if two cycles of modulation occur during an even number of natural periods.

If the changes of a parameter are produced with the above mentioned periodicity but not abruptly, the influence of these changes on the oscillator is qualitatively quite similar, though the efficiency of the parametric delivery of energy (at the same amplitude of the parametric modulation) is maximal for the square-wave time dependence, because this form of modulation provides optimal conditions for the transfer of energy to the oscillating system. The case of the sinusoidal modulation of some parameter is important for practical applications.

1.3 Electromagnetic Analogue of Mechanical System

Parametric excitation is possible in various oscillatory systems. The right-hand side of figure 1 shows an electromagnetic analogue of the spring oscillator: a series *LCR*-circuit containing a capacitor, an inductor (a coil), and a resistor. Oscillating current can be excited by periodic changes of the capacitance if we periodically move the plates closer together and farther apart, or by changes of the inductance of the coil if we periodically move an iron core in and out of the coil. Such periodic changes of the inductance are quite similar to the changes of the moment of inertia by mass redistribution in the mechanical system, which cause modulation of the natural frequency $\omega_0 = \sqrt{D/J}$ of the torsional oscillations of the rotor.

The strongest oscillations are excited when the cycle of such changes is repeated twice during one period of natural electromagnetic oscillations in the circuit, i.e., when the frequency of a parametric modulation is twice the natural frequency of the system. It is evident that parametric excitation can occur only if at least weak natural oscillations already exist in the system.

Parametric excitation is possible only with the modulation of one of the energy-conserving parameters, C or L. Modulation of the resistance R (or of the damping constant γ in the mechanical system) can affect only the character of the damping of oscillations. It cannot generate an increase in their amplitude.

1.4 Conditions for Parametric Resonance

There are several important differences that distinguish parametric resonance from the ordinary resonance caused by an external force acting directly on the system. The growth of the amplitude and hence of the energy of oscillations during parametric excitation is provided by the work of forces that periodically change the parameter. Maximal energy transfer to the oscillatory system occurs when the parameter is changed twice during one period of the excited natural oscillations. But the delivery of energy, though less efficient, is possible when the parameter changes once during one period, twice during three periods, and so on. That is, resonance is possible when one of the following conditions for the frequency ω (or for the period T) of a parameter modulation is fulfilled:

$$\omega = 2\omega_0/n, \qquad T = nT_0/2, \qquad n = 1,\ 2,\ \ldots \tag{2}$$

For a given amplitude of parametric modulation, the higher the order n of resonance, the less (in general) the amount of energy delivered to the oscillating system during one period.

One of the most interesting characteristics of parametric resonance is the possibility of exciting increasing oscillations not only at the frequencies ω_n given in Eq. (2), but also in a range of frequencies ω lying on either side of the values ω_n (in the *ranges of instability*.) These intervals become wider as the degree (depth) of modulation is increased, that is, as the range of parametric variation is extended. By this range we mean, in the case of the rotor, the difference in the maximal and minimal values of its moment of inertia, and in the oscillating electrical circuit, the differences in the inductance of the coil.

An important difference between parametric excitation and forced oscillations is related to the dependence of the growth of energy on the energy already stored in the system. While for forced excitation the increment of energy during one period is proportional to the *amplitude* of oscillations, i.e., to the square root of the energy, at parametric resonance the increment of energy is proportional to the *energy* stored in the system.

Energy losses caused by friction (unavoidable in any real system) are also proportional to the energy already stored. In the case of direct forced excitation, an arbitrarily small external force gives rise to resonance. However, energy losses restrict the growth of the amplitude because these losses grow with the energy faster than does the investment of energy arising from the work done by the external force.

In the case of parametric resonance, both the investment of energy caused by the modulation of a parameter and the frictional losses are proportional to the energy stored (to the square of the amplitude), and so their ratio does not depend on the amplitude. Therefore, parametric resonance is possible only when a *threshold* is exceeded, that is, when the increment of energy during a period (caused by the parametric variation) is larger than the amount of energy dissipated during the same time. To satisfy this requirement, the range of the parametric variation (the depth of modulation) must exceed some critical value. This critical (threshold) value of the modulation depth depends on friction. However, if the threshold is exceeded, the frictional losses of energy cannot restrict the growth of the amplitude. In a linear system the amplitude of parametrically excited oscillations must grow infinitely.

In a nonlinear system the natural period depends on the amplitude of oscillations. If conditions for parametric resonance are fulfilled at small oscillations and the amplitude begins to grow, the conditions of resonance become violated at large amplitudes. In a real system the growth of the amplitude is restricted by nonlinear effects.

1.5 Differential Equation for Parametric Oscillations

We next consider a more rigorous mathematical treatment of parametric resonance under a square-wave modulation of the parameter. This treatment is based on the differential equations governing the phenomenon.

Let the changes in the moment of inertia J of the rotor be described by a square-wave piecewise-constant dependence on time. The maximal J_1 and minimal J_2 values of the moment of inertia are equal to $J_0(1+m)$ and $J_0(1-m)$ respectively. During the time intervals $(0,\ T/2)$ and $(T/2,\ T)$, the value of the moment of inertia is constant, and the motion of the rotor can be considered as a free oscillation described by a linear differential equation. However, the coefficients in this equation are different for the adjacent time intervals $(0,\ T/2)$ and $(T/2,\ T)$:

$$\ddot{\varphi} = -\frac{1}{1+m}(\omega_0^2\varphi + 2\gamma\dot{\varphi}) \qquad \text{for} \qquad 0 < t < T/2, \tag{3}$$

$$\ddot{\varphi} = -\frac{1}{1-m}(\omega_0^2\varphi + 2\gamma\dot{\varphi}) \qquad \text{for} \qquad T/2 < t < T. \tag{4}$$

Here $\omega_0 = \sqrt{D/J_0}$ is the natural frequency of the oscillator and γ is the damping constant characterizing the strength of viscous friction. Both these quantities correspond to the mean value $J_0 = \frac{1}{2}(J_1 + J_2)$ of the moment of inertia. For small and moderate values of m the moment of inertia equals J_0 when the weights are near the half-way point between their extreme positions on the rod. For large m this is not the case because the moment of inertia depends on the square of the distance of the weights from the axis of rotation.

At the instant of an abrupt change of the moment of inertia we must make a transition from one of these linear equations to the other. Eq. (3) and Eq. (4) replace one another at $t = nT/2$, where

$n = 1,\ 2,\ \ldots$

In the simulation computer program a real time numerical integration of Eq. (3) and Eq. (4) describing the motion of the oscillator during adjacent time intervals $(0,\ T/2)$, $(T/2,\ T)$, $(T,\ 3T/2)$, etc., is performed.

The initial conditions for each subsequent time interval are chosen according to the physical model in the following way. Each initial value of the angular displacement φ equals the value $\varphi(t)$ reached by the oscillator at the end of the preceding time interval. The initial value of the angular velocity $\dot{\varphi}$ is related to the angular velocity at the end of the preceding time interval by the law of conservation of the angular momentum:

$$(1+m)\dot{\varphi}_1 = (1-m)\dot{\varphi}_2. \tag{5}$$

In Eq. (5) $\dot{\varphi}_1$ is the angular velocity at the end of the preceding time interval, when the moment of inertia of the rotor has the value $J_1 = J_0(1+m)$, and $\dot{\varphi}_2$ is the initial value for the following time interval, during which the moment of inertia is equal to $J_2 = J_0(1-m)$.

The change of the angular velocity at an abrupt variation of the inertia moment from the value J_2 to J_1 can be found in the same way.

That we may use the conservation of angular momentum, as expressed in Eq. (5), is allowed because, at sufficiently rapid displacement of the weights along the rotor, we can neglect the influence of the spring and consider the rotor as if it were freely rotating about its axis. This assumption is valid provided the duration of the displacement of the weights is a small part of the natural period (of the period of free oscillations).

During each half-period $T/2$ the motion of the oscillator is described by a linear differential equation, Eq. (3) or Eq. (4). That is, this motion is a segment of some harmonic (or damped) natural oscillation. An analytical investigation of parametric excitation can be carried out by fitting to one another known solutions of the linear equations for consecutive adjacent time intervals.

An analytical investigation of the problem is usually restricted to the determination of the ranges of the frequency ω and the depth m of modulation, in which the state of rest in the equilibrium position becomes unstable. In these *ranges of instability* an arbitrarily small deflection from equilibrium is sufficient for the progressive growth of small initial oscillations. In other words, for every given value of the modulation depth m, we can determine the intervals of frequencies ω near the values $\omega_n = 2\omega_0/n$ in which parametric resonance (i.e., oscillation with increasing amplitude) is possible. We can do this considering the conditions for which Eqs. (3)–(4) yield solutions with increasing amplitudes.

1.6 The Threshold of Parametric Excitation at Square-Wave Modulation

We can use arguments employing the conservation of energy to evaluate the modulation depth which corresponds to the threshold of parametric excitation. First, let us find the increment of the rotor kinetic energy which occurs during an abrupt shift of the weights toward the axis, when the moment of inertia decreases from the value $J_1 = J_0(1+m)$ to the value $J_2 = J_0(1-m)$. We consider the case of small values m of the modulation depth ($m \ll 1$). During radial displacements of the weights along the rod, the angular momentum $L = J\omega$ of the rotor is conserved. Therefore it is convenient to use the expression $E_{\text{kin}} = L^2/(2J)$, which gives the kinetic energy of the rotor in terms of L. For the increment ΔE of this kinetic energy we can write:

$$\Delta E = \frac{L^2}{2}\left(\frac{1}{J_2} - \frac{1}{J_1}\right) = \frac{L^2}{2J_0}\left(\frac{1}{1-m} - \frac{1}{1+m}\right) \approx 2m\frac{L^2}{2J_0} \approx 2mE_{\text{kin}}. \tag{6}$$

If the event occurs near the equilibrium position of the rotor, when the total energy E of the pendulum is approximately its kinetic energy $E_{\rm kin}$, we see from Eq. (6) that the relative increment of the total energy $\Delta E/E$ approximately equals twice the value of the modulation depth m: $\Delta E/E \approx 2m$.

When the frequencies and phases have those values which are favorable for the most effective delivery of energy, the abrupt displacement of the weights toward the ends of the rod occurs at the instant when the rotor attains its greatest deflection (more precisely, when the rotor is very near it). At this instant the angular velocity of the rotor is almost zero, and so this radial displacement of the weights into their previous positions causes no decrement of the energy.

For the principal resonance ($n = 1$) the investment of energy occurs twice during the natural period T_0 of oscillations. That is, the relative increment of energy $\Delta E/E$ during one period approximately equals $4m$. A process in which the increment of energy ΔE during a period is proportional to the energy stored E ($\Delta E \approx 4mE$) is characterized by the exponential growth of the energy in time:

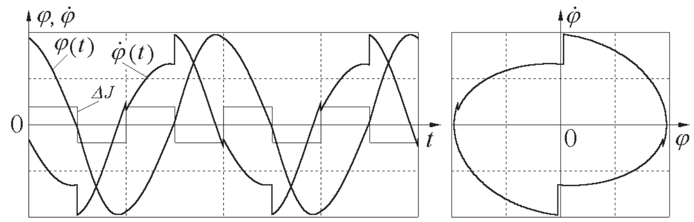
$$E(t) = E_0 \exp(\alpha t). \tag{7}$$

In this case the index of growth α is proportional to the depth of modulation m of the moment of inertia: $\alpha = 4m/T_0$. When the modulation is exactly tuned to the principal resonance ($T = T_0/2$), the decrease of energy is caused only by friction. Dissipation of energy due to viscous friction during an integral number of cycles is described by the following expression:

$$E(t) = E_0 \exp(-2\gamma t). \tag{8}$$

Equation (8) yields the relative decrease of energy $\Delta E/E$ during a time interval t containing an integral number of natural periods: $\Delta E/E \approx -2\gamma t$. Equating the relative increment $4m$ of energy during one period (caused by the square-wave parameter modulation) to the relative energy losses due to friction $2\gamma T_0$, we obtain the following estimate for the threshold (minimal) value $m_{\rm min}$ of the depth of modulation corresponding to the excitation of the principal parametric resonance:

$$m_{\rm min} = \gamma T_0/2 = \pi/(2Q). \tag{9}$$



Period of modulation 0.5, depth of modulation 20%, quality factor 8.29, initial deflection 20 degrees, initial angular velocity -0.07

Figure 2: The time-dependent graphs and the phase trajectory of stationary oscillations at the threshold condition $m \approx \pi/2Q$ for $T = T_0/2$.

The plot of the angular velocity and the phase trajectory of parametric oscillations occurring at the threshold conditions, equation (9), are shown in figure 2. This mode of steady oscillations (which have a constant amplitude in spite of the dissipation of energy) is called *parametric regeneration*. The stationary character of such oscillations is possible because frictional losses of the energy are on the average compensated for by the energy delivery from the source that makes the weights move along the rod and thus causes the periodic changes in the moment of inertia.

For the resonance for which $T = 3T_0/2$ the threshold value of the depth of modulation is three times greater than its value for the principal resonance: $m_{\min} = 3\pi/(2Q)$. In this instance two cycles of the parametric variation occur during three full periods of natural oscillations. Radial displacements of the weights again happen at the most favorable moments, and so the same investment of energy occurs during an interval that is three times longer than the interval for the principal resonance.

When the depth of modulation exceeds the threshold value, the energy of oscillations increases exponentially in time. The growth of the energy again is described by Eq. (7). However, now the index of growth α is determined by the amount by which the energy delivered through parametric modulation exceeds the simultaneous losses of energy caused by friction: $\alpha = 4m/T_0 - 2\gamma$. The energy of oscillations is proportional to the square of the amplitude. Therefore the amplitude of parametrically excited oscillations also increases exponentially with time (figure 3): $a(t) = a_0 \exp(\beta t)$. The index β in the growth of amplitude is one half the index of the growth in energy. For the principal resonance, when the investment of energy occurs twice during one natural period of oscillation, we have $\beta = 2m/T_0 - \gamma = m\omega_0/\pi - \gamma$.

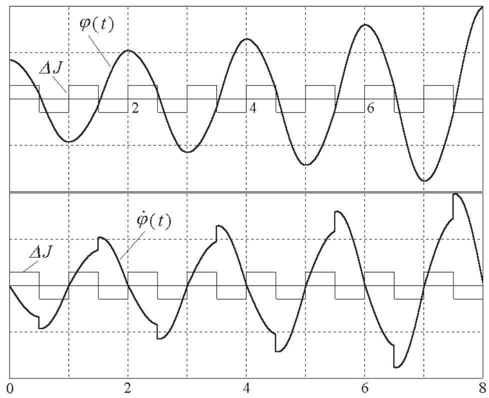


Figure 3: Exponential growth of the amplitude of oscillations at parametric resonance of the first order ($n = 1$).

1.7 Frequency Ranges of Parametric Excitation

The threshold for the parametric excitation of the torsion pendulum is determined above for the resonant situations in which two cycles of the parametric modulation occur during one natural period or during three natural periods of oscillation. The estimate obtained, Eq. (9), is valid for small values of the modulation depth m.

For large values of the modulation depth m, the notion of a natural period needs a more precise definition. Let $T_0 = 2\pi/\omega_0 = 2\pi\sqrt{J_0/D}$ be the period of oscillation of the rotor when the weights are fixed in their middle positions. The corresponding moment of inertia equals $J_0 = \frac{1}{2}(J_{\max} + J_{\min})$. The period is somewhat longer when the weights are moved further apart. It has the value $T_1 = T_0\sqrt{1+m} \approx T_0(1+m/2)$. The period is shorter when the weights are moved closer to one another: $T_2 = T_0\sqrt{1-m} \approx T_0(1-m/2)$.

It is convenient to define the average period T_{av} not as the arithmetic mean $\frac{1}{2}(T_1 + T_2)$, but rather as the period that corresponds to the arithmetic mean frequency $\omega_{\text{av}} = \frac{1}{2}(\omega_1 + \omega_2)$, where $\omega_1 = 2\pi/T_1$ and $\omega_2 = 2\pi/T_2$. So we define T_{av} by the relation:

$$T_{\text{av}} = \frac{2\pi}{\omega_{\text{av}}} = \frac{2T_1T_2}{(T_1 + T_2)}. \tag{10}$$

The period T of the parametric modulation which is exactly tuned to any of the parametric resonances is determined not only by the order n of the resonance, but also by the depth of modulation m. In order to satisfy the resonant conditions, the increment in the phase of natural oscillations during one cycle of modulation must be equal to π, 2π, 3π, $\dots$, $n\pi$, $\dots$ During the first half-cycle the phase increases by $\omega_1 T/2$, and during the second half-cycle—by $\omega_2 T/2$. Consequently, instead of the approximate condition expressed by Eq. (2), we obtain:

$$\frac{\omega_1+\omega_2}{2}T = n\pi, \qquad \text{or} \qquad T = n\frac{\pi}{\omega_{\mathrm{av}}} = n\frac{T_{\mathrm{av}}}{2}. \tag{11}$$

Thus, for a parametric resonance of some definite order n, the condition for exact tuning can be expressed in terms of the two natural periods, T_1 and T_2. This condition is $T = nT_{\mathrm{av}}/2$, where T_{av} is defined by Eq. (10).

For moderate values of m it is possible to use an approximate expression for the average frequency and period:

$$\omega_{\mathrm{av}} = \frac{\omega_0}{2}\left(\frac{1}{\sqrt{1+m}}+\frac{1}{\sqrt{1-m}}\right) \approx \omega_0\left(1+\frac{3}{8}m^2\right), \quad T_{\mathrm{av}} \approx T_0\left(1-\frac{3}{8}m^2\right).$$

The difference between T_{av} and T_0 reveals itself in terms proportional to the square of the depth of modulation m.

An infinite growth of the amplitude during parametric excitation is possible not only at exact tuning to one of resonances but in certain *intervals* of T-values. These intervals, or the *ranges of instability,* surround the resonant values $T = T_{\mathrm{av}}/2$, $T = T_{\mathrm{av}}$, $T = 3T_{\mathrm{av}}/2, \dots$ The width of the intervals increases with the depth m of the parameter modulation.[1] Outside the intervals the equilibrium position of a torsion pendulum is stable, and the amplitude of oscillations does not grow.

In order to determine the boundaries of the frequency ranges of instability, we can consider *stationary oscillations* that occur when the period of modulation T equals that of one of the boundaries. These stationary oscillations can be represented as an alternation of free oscillations with the periods T_1 and T_2. In the absence of friction the graphs of such oscillations are formed by segments of non-damped sine curves with the corresponding periods.

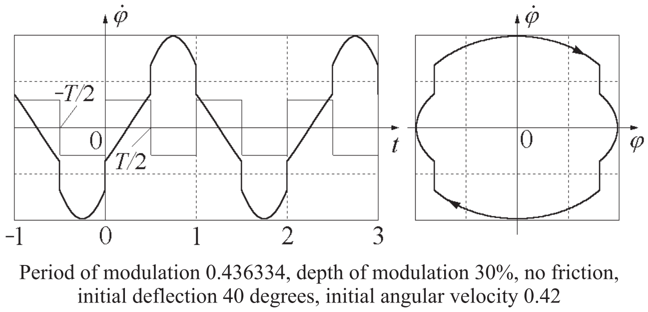


Figure 4: Stationary parametric oscillations at the lower boundary of the principal interval of instability near ($T = T_{\mathrm{av}}/2$).

We examine first the vicinity of the principal resonance occurring at $T = T_{\mathrm{av}}/2$. Suppose that the period T of the parametric variation is a little shorter than the resonant value $T = T_{\mathrm{av}}/2$, so that T corresponds to the left boundary of the interval of instability. In this case a little less than a quarter of the

[1]Strictly speaking, for high resonances ($n > 2$) this statement is true only for small and moderate values of the depth of modulation (see figure 10 with the diagram of the ranges of parametric resonance).

mean natural period $T_{\rm av}$ elapses between consecutive abrupt increases and decreases of the moment of inertia. The graph of the angular velocity $\dot\varphi(t)$ for this periodic stationary process has the characteristic pattern shown in figure 4. The segments of the graphs of free oscillations (which occur during time intervals during which the moment of inertia is constant) are alternating parts of sine or cosine curves with the periods T_1 and T_2. These segments are symmetrically truncated on both sides.

To find conditions at which such stationary oscillations take place, we can write the expressions for $\varphi(t)$ and $\dot\varphi(t)$ during the adjacent intervals in which the oscillator executes natural oscillations, and then fit these expressions to one another at the boundaries. Such fitting must provide a periodic stationary process.

We let the origin of time, $t = 0$, be the instant when the weights are shifted apart. The angular velocity is abruptly decreased in magnitude at this instant (see figure 4). Then during the interval $(0,\ T/2)$ the graph describes a natural oscillation with the frequency $\omega_1 = \omega_0/\sqrt{1+m}$. It is convenient to represent this motion as a superposition of sine and cosine waves whose constant amplitudes are A_1 and B_1:

$$\begin{aligned}\varphi_1(t) &= (A_1 \sin\omega_1 t + B_1 \cos\omega_1 t),\\ \dot\varphi_1(t) &= (A_1\omega_1 \cos\omega_1 t - B_1\omega_1 \sin\omega_1 t).\end{aligned} \tag{12}$$

Similarly, during the interval $(-T/2,\ 0)$ the graph in figure 4 is a segment of natural oscillation with the frequency $\omega_2 = \omega_0/\sqrt{1-m}$:

$$\begin{aligned}\varphi_2(t) &= (A_2 \sin\omega_2 t + B_2 \cos\omega_2 t),\\ \dot\varphi_2(t) &= (A_2\omega_2 \cos\omega_2 t - B_2\omega_2 \sin\omega_2 t).\end{aligned} \tag{13}$$

To determine the values of constants A_1, B_1 and A_2, B_2, we can use the conditions that must be satisfied when the segments of the graph are joined together, taking into account the periodicity of the stationary process.

At $t = 0$ the angle of deflection is the same for both φ_1 and φ_2: $\varphi_1(0) = \varphi_2(0)$. From this condition we find that $B_1 = B_2$. We later denote these equal constants simply by B. The angular velocity at $t = 0$ undergoes a sudden change:

$$(1+m)\dot\varphi_1(0) = (1-m)\dot\varphi_2(0).$$

This condition gives us the following relation between A_2 and A_1: $A_2 = kA_1 = kA$, where we have introduced a dimensionless quantity k which depends on the depth of modulation m:

$$k = \sqrt{\frac{1+m}{1-m}}.$$

Equations for the constants A and B are determined by the conditions at the instants $-T/2$ and $T/2$. For stationary periodic oscillations, corresponding to the principal resonance (and to all resonances of odd numbers $n = 1,\ 3,\ \dots$ in Eq. (11)), these conditions are:

$$\varphi_1(T/2) = -\varphi_2(-T/2), \qquad (1+m)\dot\varphi_1(T/2) = -(1-m)\dot\varphi_2(-T/2). \tag{14}$$

Substituting φ and $\dot\varphi$ from Eq. (13) in Eq. (14), we obtain the system of homogeneous equations for the unknown quantities A and B:

$$\begin{aligned}(S_1 - kS_2)A + (C_1 + C_2)B &= 0,\\ k(C_1 + C_2)A - (kS_1 - S_2)B &= 0.\end{aligned} \tag{15}$$

In Eq. (15) the following notations are used:

$$C_1 = \cos(\omega_1 T/2), \quad C_2 = \cos(\omega_2 T/2),$$
$$S_1 = \sin(\omega_1 T/2), \quad S_2 = \sin(\omega_2 T/2). \tag{16}$$

The homogeneous system of equations for A and B, Eq. (15), has a non-trivial (non-zero) solution only if its determinant is zero:

$$2kC_1C_2 - (1+k^2)S_1S_2 + 2k = 0. \tag{17}$$

This condition for the existence of a non-zero solution to Eq. (15) gives us an equation for the unknown variable T, which enters in Eq. (17) as the arguments of sine and cosine functions in S_1, S_2 and C_1, C_2. This equation determines the boundaries of the interval of instability. These boundaries T_- and T_+ are given by the roots of the equation.

To find approximate solutions T to this transcendental equation, Eq. (17), we transform it into a more convenient form. We first represent in Eq. (17) the products C_1C_2 and S_1S_2 as follows:

$$C_1C_2 = \frac{1}{2}(\cos\frac{\Delta\omega T}{2} + \cos\omega_{\rm av}T), \qquad S_1S_2 = \frac{1}{2}(\cos\frac{\Delta\omega T}{2} - \cos\omega_{\rm av}T),$$

where $\Delta\omega = \omega_2 - \omega_1$. Then, using the identity $\cos\alpha = 2\cos^2(\alpha/2) - 1$, we reduce Eq. (17) to the following form:

$$(1+k)\cos\frac{\omega_{\rm av}T}{2} = \pm|1-k|\cos\frac{\Delta\omega T}{4}. \tag{18}$$

For the boundaries of the instability interval which contains the principal resonance, we search for a solution T of Eq. (18) in the vicinity of $T = T_0/2$. For a given value of the depth of modulation m, Eq. (18) in the neighborhood of $T_0/2 \approx T_{\rm av}/2$ has two solutions which correspond to the boundaries T_- and T_+ of the instability interval. The phase diagram and the graph of the angular velocity for the right boundary of the main interval ($n = 1$) are shown in figure 5.

To find the boundaries T_- and T_+ of the instability interval, we replace T in the argument of the cosine on the left-hand side of Eq. (18) by $T_{\rm av}/2 + \Delta T$, where $\Delta T \ll T_0$. Since $\omega_{\rm av}T_{\rm av} = 2\pi$, we can write the cosine as $-\sin(\omega_{\rm av}\Delta T/2)$. Then Eq. (18) becomes:

$$\sin\frac{\omega_{\rm av}\Delta T}{2} = \mp\frac{|1-k|}{1+k}\sin\frac{\Delta\omega(T_{\rm av}/2 + \Delta T)}{4}. \tag{19}$$

This equation for ΔT can be solved numerically by iteration. We start with $\Delta T = 0$ as an approximation of the zeroth order, substituting it into the right-hand side of Eq. (19), taken, say, with the upper sign. Then the left-hand side of Eq. (19) gives us the value of ΔT to the first order. We substitute this first-order value into the right-hand side of Eq. (19), and on the left-hand side we obtain ΔT to the second order. This procedure is iterated until a self-consistent value of ΔT for the left boundary is obtained. To determine ΔT for the right boundary, we can use the same procedure, taking the lower sign on the right-hand side of Eq. (19).

After the substitution of one of the roots T_- or T_+ of Eq. (19) into Eqs. (15) both equations for A and B become equivalent and permit us to find only the ratio A/B. This limitation means that the amplitude of stationary oscillations at the boundary of the instability interval can be arbitrarily large. This amplitude depends on the initial conditions. Nevertheless, these oscillations have a definite shape which is determined by the ratio of the amplitudes A and B of the sine and cosine functions whose segments form the pattern of the stationary parametric oscillation (see figures 4 and 5).

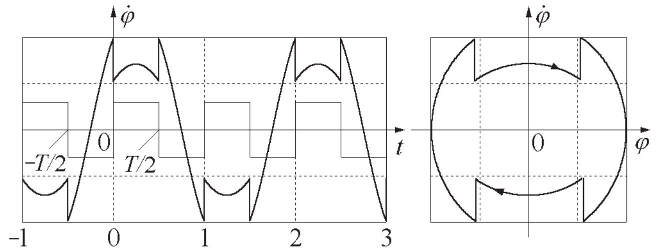


Figure 5: Stationary parametric oscillations at the upper boundary of the principal interval of instability (near $T = T_{\text{av}}/2$).

To obtain an approximate analytic solution to Eq. (19) that is valid for small values of the modulation depth m, we can simplify the expression on the right-hand side by assuming that $k \approx 1+m$, $|1-k| \approx m$. We may also assume the value of the cosine to be 1. On the left-hand side of Eq. (19), the sine can be replaced by its small argument, where $\omega_{\text{av}} = 2\pi/T_{\text{av}}$. Thus we obtain the following approximate expression that is valid up to terms to the second order in m:

$$T_{\mp} = \frac{1}{2}\left(1 \mp \frac{m}{\pi}\right) T_{\text{av}}. \tag{20}$$

Since the natural period $T_0 = 2\pi\sqrt{D/J_0}$ is used in the program as an appropriate time unit for the input of the period of modulation T, we express the values of $T_{\mp}$ given by Eq. (20) in terms of T_0:

$$T_{\mp} = \frac{1}{2}\left(1 \mp \frac{m}{\pi} - \frac{3m^2}{8}\right) T_0. \tag{21}$$

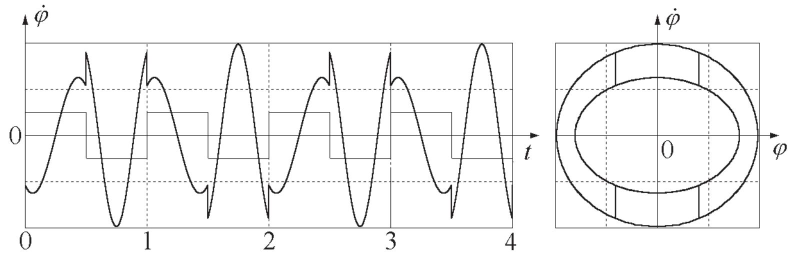


Figure 6: Stationary parametric oscillations at one of the boundaries of the interval of instability near $T = 3T_{\text{av}}/2$.

In a similar way we can determine the boundaries of the instability interval in the vicinity of a resonance of the higher order $n = 3$. At this resonance two cycles of the parametric variation occur during three natural periods of oscillation ($T = 3T_{\text{av}}/2$). Considering stationary oscillations at the boundaries of the interval (figure 6), we get the same equations, Eqs. (15), for A and B, as well as Eq. (18) for the values of the period of modulation. However, now we should search for a solution to Eqs. (15) in the vicinity of $T = 3T_{\text{av}}/2$. The boundaries of this interval, obtained by a numerical solution, are also displayed on the screen after we input the depth of modulation m.

For small values of the depth of modulation m, we can find approximate analytic expressions for the lower and the upper boundaries of the interval that are valid up to quadratic terms in m:

$$T_{\mp} = \left(\frac{3}{2} \mp \frac{m}{2\pi}\right) T_{\rm av}. \tag{22}$$

In terms of the mean natural period T_0 these boundaries are expressed as follows:

$$T_{\mp} = \left(\frac{3}{2} \mp \frac{m}{2\pi} - \frac{9m^2}{16}\right) T_0. \tag{23}$$

In this approximation, the third interval has the same width $(m/\pi)T_0$ as does the interval of instability in the vicinity of the principal resonance. However, this interval is distinguished by greater asymmetry: its central point is displaced to the left of the value $T = 3T_0/2$ by $(9/16)m^2T_0$.

For moderate square-wave modulation of the moment of inertia, parametric resonance of the order $n = 2$ (one cycle of the parametric variation during one natural period of oscillation) is relatively weak compared to the resonances $n = 1$ and $n = 3$ considered above. In the case in which $n = 2$ the abrupt changes of the moment of inertia induce both an increase and a decrease of the energy only once during each natural period. The growth of oscillations occurs only if the increase in energy at the instant when the weights are drawn closer is greater than the decrease in energy when the weights are drawn apart. This is possible only if the weights are shifted toward the axis when the angular velocity of the rotor is greater in magnitude than it is when they are shifted apart. For $T \approx T_{\rm av}$, these conditions can fulfill only because there is a small difference between the natural periods T_1 and T_2 of the rotor, where T_1 is the period with the weights shifted apart and T_2 is the period with them shifted together. This difference is proportional to m.

The growth of oscillations at parametric resonance of the second order is shown in figure 7. In this case, the investment of energy during a period is proportional to the *square* of the depth of modulation m, while in the cases of resonances with $n = 1$ and $n = 3$ the investment of energy is proportional to the first power of m. Therefore, for the same value of the damping constant γ (the same quality factor Q), a considerably greater depth of modulation is required here to exceed the threshold of parametric excitation.

The interval of instability in the vicinity of resonance with $n = 2$ is considerably narrower compared to the corresponding intervals of the resonances with $n = 1$ and $n = 3$. Its width is also proportional only to the square of m (for small values of m).

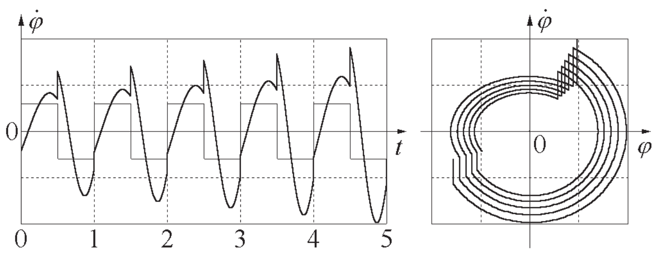


Figure 7: The graph of the angular velocity and the phase trajectory of oscillations corresponding to parametric resonance of the second order ($T = T_{\rm av}$).

To determine the boundaries of this interval of instability, we can consider, as is done above for other resonances, stationary oscillations for $T \approx T_0$ formed by alternating segments of free sinusoidal oscillations with the periods T_1 and T_2. The phase trajectory and the graph of the angular velocity of such stationary periodic oscillations for one of the boundaries are shown in figure 8. During oscillations occurring at the boundary of the instability interval, the abrupt increment and decrement of the angular velocity exactly compensate each other.

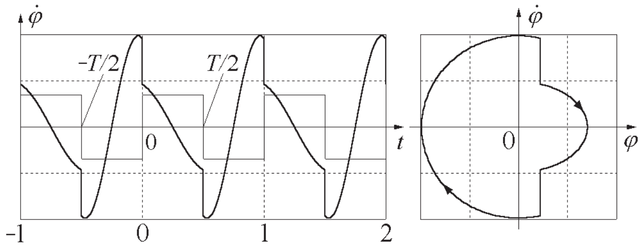


Figure 8: Stationary parametric oscillations at one of the boundaries of the interval of instability of the second order (near $T = T_{\rm av} \approx T_0$).

To describe these stationary oscillations, we can use the same expressions for $\varphi(t)$ and $\dot\varphi(t)$ as we use in Eqs. (12)–(13). The conditions for joining the graphs at $t = 0$ are also the same. However, differences begin with the equations for the constants A and B. They are determined by the conditions of periodicity at the instants $-T/2$ and $T/2$. For stationary periodic oscillations, corresponding to resonance with $n = 2$ (and for all resonances of even orders $n = 2,\ 4,\ \dots$ in Eq. (11)), these conditions are:

$$\varphi_1(T/2) = \varphi_2(-T/2), \qquad (1+m)\dot\varphi_1(T/2) = (1-m)\dot\varphi_2(-T/2), \tag{24}$$

and we obtain the system of equations for the amplitudes A and B:

$$\begin{aligned} &(S_1 + kS_2)A + (C_1 - C_2)B = 0,\\ &k(C_1 - C_2)A - (kS_1 + S_2)B = 0, \end{aligned} \tag{25}$$

where S_1, C_1 and S_2, C_2 are defined by the same Eqs. (16). The homogeneous system of equations for A and B, Eqs. (25), has a non-trivial solution if its determinant is zero:

$$2kC_1C_2 - (1+k^2)S_1S_2 - 2k = 0. \tag{26}$$

In order to find the values $T_\mp = T_{\rm av} + \Delta T$ for the instability interval with $n = 2$ from Eq. (26), we transform the products C_1C_2 and S_1S_2 in Eq. (26) by using the identity $\cos\alpha = 1 - 2\sin^2(\alpha/2)$:

$$(1+k)\sin\frac{\omega_{\rm av}T}{2} = \pm|1-k|\sin\frac{\Delta\omega T}{4}. \tag{27}$$

We next replace T in the argument of the sine on the left-hand side of Eq. (27) by $T_{\rm av} + \Delta T$, where $\Delta T \ll T_0$. Since $\omega_{\rm av}T_{\rm av} = 2\pi$, we can write this sine as $-\sin(\omega_{\rm av}\Delta T/2)$. Then Eq. (18) becomes:

$$\sin\frac{\omega_{\rm av}\Delta T}{2} = \mp\frac{|1-k|}{1+k}\cos\frac{\Delta\omega(T_{\rm av}+\Delta T)}{4}. \tag{28}$$

This equation gives the left boundary T_- of the instability interval when we take the upper sign in its right-hand side, and the right boundary T_+ when we take the lower sign. Stationary oscillations, which correspond to the right boundary, are shown in figure 9.

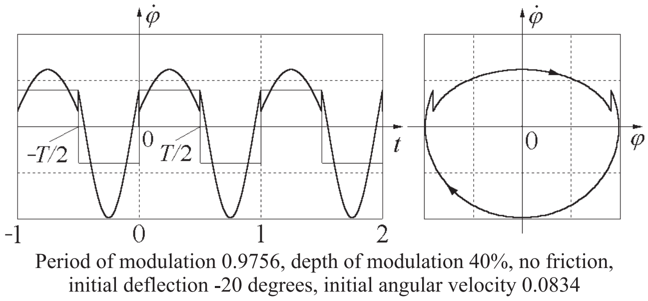


Figure 9: Stationary parametric oscillations at the other boundary of the interval of instability of the second order (near $T = T_{\rm av} \approx T_0$).

In the simulation program Eq. (28) for ΔT is also solved numerically by iteration. Substituting T_- or T_+ obtained from (28) into one of the Eqs. (15), we get the ratio of the amplitudes A and B that determines the pattern of stationary oscillations at the corresponding boundary of the instability interval.

For moderate values of the depth of modulation, it is possible to find an approximate analytical solution of Eq. (28):

$$T_{\mp} = \left(1 \mp \frac{1}{4}m^2\right) T_{\rm av}. \qquad (29)$$

In terms of T_0 the boundaries of the second interval are:

$$T_{\mp} = T_0 + \left(\mp\frac{1}{4} - \frac{3}{8}\right) m^2 T_0, \qquad (30)$$

i.e., $T_- = T_0 - (5/8)m^2T_0$, $T_+ = T_0 - (1/8)m^2T_0$. As mentioned above, the width $T_+ - T_- = (m^2/2)T_0$ of this interval of instability is proportional to the square of the modulation depth.

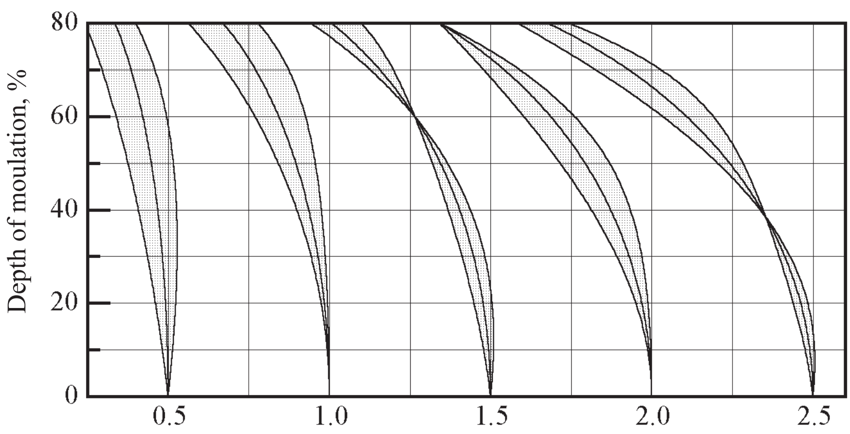
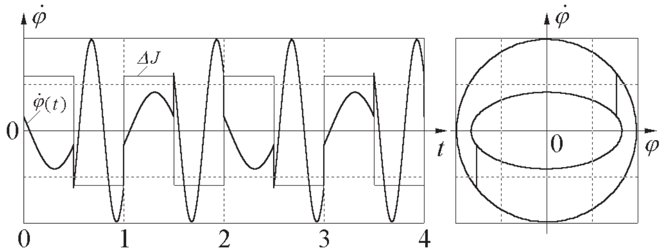


Figure 10: Intervals of the parametric excitation at square-wave modulation of the inertia moment in the absence of friction.

The intervals of instability for the first five parametric resonances are shown in the diagram (figure 10) for various values of the modulation depth m. The diagram is obtained by numerical solution of the equations which are discussed above. We note how narrow the intervals of even resonances (n =2, 4) are for small values of m. With the growth of m the intervals expand and become comparable with the intervals of odd orders.

Figure 10 shows that at some definite values of m both boundaries of intervals with $n > 2$ coincide (we may consider that they *intersect*). Thus at these values of m the corresponding intervals of parametric resonance disappear. These values of m correspond to the natural periods T_1 and T_2 of oscillation (associated with the weights far apart and close to each other), whose ratio is $2 : 1$, $3 : 1$, and $3 : 2$.



Period of modulation 1.264911, depth of modulation 60%, no friction, initial deflection 90 degrees, initial angular velocity 0.5

Figure 11: Time-dependent graph of the angular velocity and the phase trajectory for stationary oscillations at the intersection of both boundaries of the third interval.

For the first intersection (ratio $2 : 1$) exactly one half of the natural oscillation with period T_1 is completed during the first half of the modulation cycle (see figure 11). On the phase diagram, the representing point traces a half of the smaller ellipse, and then abruptly jumps down to the larger ellipse. During the second half of the modulation cycle the oscillator executes exactly a whole natural oscillation with period $T_2 = T_1/2$, so that the representing point passes in the phase plane along the whole larger ellipse, and then jumps up to the smaller ellipse along the same vertical segment.

During the next modulation cycle the representing point generates first the other half of the smaller ellipse, and then again the whole larger ellipse. Therefore during any two adjacent cycles of modulation the representing point passes once along the closed smaller ellipse and twice along the larger one, returning finally to the initial point of the phase plane. We see that such an oscillation is periodic for arbitrary initial conditions. This means that for the corresponding values of the modulation depth m and the period of modulation T the growth of amplitude is impossible even in the absence of friction (the instability interval vanishes).

Similar explanations can be suggested for other cases in figure 10 in which the boundaries intersect.

When there is friction in the system, the intervals of the period of modulation become narrower, and for strong enough friction (below the threshold) the intervals disappear. Above the threshold, approximate values for the boundaries of the first interval are given by Eq. (20) or Eq. (21) provided we substitute for m the expression $\sqrt{m^2 - m_{\min}^2}$ with the threshold value $m_{\min} = \pi/(2Q)$ defined by Eq. (9). The proof is left as an exercise (Problem 1.9). For the third interval, we can use Eq. (22) or Eq. (22), substituting $\sqrt{m^2 - m_{\min}^2}$ for m, with $m_{\min} = 3\pi/(2Q)$. When m is equal to the corresponding threshold value $m_{\min}$, the interval of parametric resonance disappears.

The boundaries of the second interval of parametric resonance in the presence of friction are given by Eq. (29) or Eq. (30) provided we substitute for m^2 the expression $\sqrt{m^4 - m_{\min}^4}$ with the threshold value $m_{\min} = \sqrt{2/Q}$, which corresponds to the second parametric resonance (see Problems 2.2 and 2.3).

The diagram in figure 12 shows the boundaries of the first three intervals of parametric resonance: in the absence of friction, for $Q = 20$, and for $Q = 10$. Note the "island" of parametric resonance for $n = 3$ and $Q = 20$. This resonance disappears when the depth of modulation exceeds 45% and reappears when m exceeds approximately 66%.

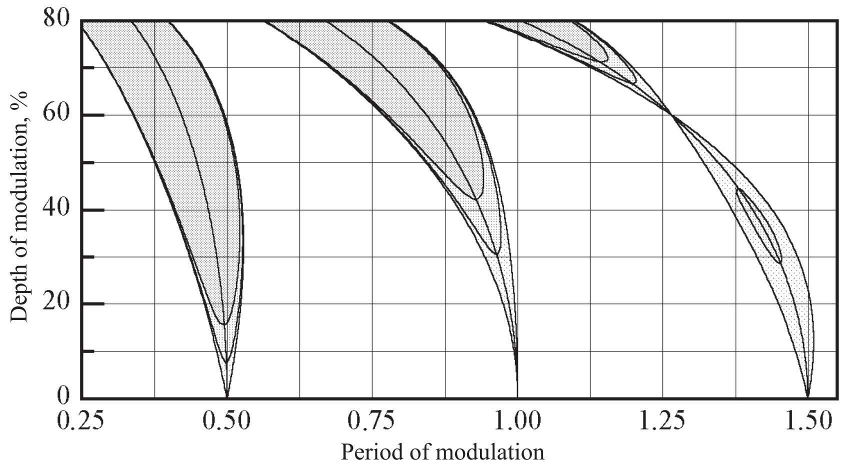


Figure 12: Intervals of parametric excitation at square-wave modulation of the moment of inertia without friction, for Q =20, and for Q =10.

For any given value m of the depth of modulation, only several first intervals of parametric resonance (where m exceeds the threshold) can exist.

We note that even if the equilibrium of the system is unstable due to modulation of the parameter (that is, if the conditions of parametric excitation are fulfilled), when the initial values of φ and $\dot{\varphi}$ are zero, they remain zero over the course of time. This behavior is in contrast to that of resonance arising from forced oscillations. In the latter instance, the amplitude increases in time even if the initial conditions are zero. In other words, if parametric resonance is to be excited, the system must already be oscillating, at least slightly, when the parametric variation first occurs.

In a linear system, if the threshold of parametric excitation is exceeded, the growth of oscillations is exponential in time. In contrast to forced oscillations, linear viscous friction is unable to restrict the growth of the amplitude at parametric resonance. In real systems the growth of the amplitude is restricted by nonlinear effects that cause the period to depend on the amplitude. During parametric excitation the growth of the amplitude changes the natural period and thereby violates the conditions of resonance.

2 Questions, Problems, Suggestions

2.1 Principal Parametric Resonance

1.1* **Principal Resonance** $(n = 1)$ **in the Absence of Friction.**

(a) Input a moderate value of the depth m of modulation of the moment of inertia (about 10–15 percent). Choose the period of modulation T to be equal to one half the mean natural period of the oscillator. What kind of initial conditions ought you to enter in order to generate from the very beginning the fastest growth of the amplitude? Remember that at the initial moment, $t = 0$, the weights are suddenly moved away from one another, further from the axis of rotation.

(b) What initial conditions would lead at first to a fading away of oscillations that are already present? Using the plots of the oscillations, explain the physical reason for the increase or decrease in amplitude. Take into account the phase relationship between the natural oscillation of the rotor and the periodic changes in its moment of inertia. Why is it that some time later a phase relation is established that generates a growth in the amplitude?

(c) Try to understand the reasons that determine the lapse of time between the initial fading of the amplitude and its subsequent infinite growth.

1.2* **The Amplitude Growth at the Principal Resonance without Friction.**

(a) If the modulation is to generate the principal parametric resonance, what rule governs the growth of the amplitude when there is an initial deflection and an initial angular velocity of zero? Calculate the depth of modulation m that, in the absence of friction, generates a doubling of the amplitude after 10 cycles of the parametric modulation. Verify your result with a simulation experiment on the computer.

(b) What difference do you find in your observations of part (a) if you set the initial deflection to be opposite the deflection in part (a)?

1.3* **The Threshold for Principal Resonance.**

(a) Choosing a moderate value for the modulation depth (say, $m = 0.15$), estimate the threshold (minimal) value of the quality factor $Q_{\min}$ that corresponds to stationary oscillations (i.e., to parametric regeneration) when the modulation is tuned to the principal resonance ($T = T_0/2$).

(b) Make your calculated estimation of the threshold value $Q_{\min}$ more exact by using an experiment on the computer. Describe the character of the plots and of the phase trajectory under conditions of parametric regeneration and explain their features.

(c) Is the mode of stationary oscillations at the threshold (for $Q = Q_{\min}$) stable with respect to small deviations in the properties of the system? Is the mode stable with respect to small deviations in the initial conditions?

(d)** The threshold value of the quality factor for any given modulation depth m is absolutely minimal when the modulation is *exactly* tuned to resonance. For small values of m the principal resonance occurs when $T = T_0/2$. However, when m increases, the resonant value of the modulation period T departs from $T_0/2$. Find this resonant value of T for an arbitrarily large modulation depth m and estimate values of T for m = 15% and m = 40%.

1.4* **The Amplitude Growth over the Threshold.**

(a) For the case in which $T = T_0/2$ and $m = 15\%$, by what factor does the amplitude of oscillation increase during 10 cycles of parametric oscillation if $Q = 2Q_{\min}$? Does the answer depend on the initial conditions? Verify your answer with a simulation experiment.

(b) What is the amplitude of oscillation after the next 10 cycles of modulation? Why does friction not restrict the growth of the amplitude of parametrically excited oscillations?

1.5** **The Principal Interval of Parametric Resonance without Friction.**

(a) Calculate the values of the period of modulation T corresponding to the boundaries of the instability interval at a given modulation depth m (in the approximation $m \ll 1$) for the case when friction is absent.

(b) How does the width of the interval depend on the depth of modulation? Do the terms of second order influence the width of the interval?

1.6** **Oscillations at a Boundary of the Instability Interval.**

(a) Enter a value of the modulation period T which corresponds to one of the boundaries of the instability interval. Remember that at these boundaries stationary oscillations of constant amplitude are possible (parametric regeneration). If you then enter initial conditions arbitrarily, the amplitude of oscillations at first grows or decreases, and the shape of oscillations differs from that of the plots in figures 4—5. Why?

(b) Observe how the pattern of oscillations gradually approaches the shape that you should expect for the chosen boundary of the interval of instability. The oscillations preserve this shape for some time, but then the amplitude begins to grow or to decrease again, and the shape of oscillations changes again. Why?

1.7* **The Initial Conditions for Steady Oscillations.**

(a) Enter the value of the period of modulation corresponding to the left boundary of the instability interval at a given value m of the modulation depth. Choose the absence of friction. Input some initial deflection. What value of the initial angular velocity ought you to enter for a given angular deflection in order that stationary oscillations of a constant amplitude occur from the beginning of the modulation?

(b) Verify your calculated approximate values of T for either boundary by simulating an experiment, and find more precise values. Explain the appearance of characteristic features of the plots and the phase trajectories of stationary oscillations corresponding to each boundary of the instability interval.

(c) For a given value of the initial displacement φ_0, and for the calculated value $\dot{\varphi}(0)$ of the initial angular velocity which provides stationary oscillations (at each of the boundaries of the interval of instability), calculate the amplitude of these oscillations. Verify the theoretical value by the experiment.

1.8** **The Threshold of Excitation within the Instability Interval.**

(a) Choose a value T of the period of modulation somewhere between the limits of the interval of instability, e.g., approximately half way between the resonant value and one of the boundaries. Evaluate experimentally the growth of the amplitude in the absence of friction, and from your observations, calculate the threshold value of the quality factor $Q = Q_{\min}$ for parametric excitation at the given value T of the modulation period.

(b) Verify your result experimentally and use the experiment to find a more exact value of $Q_{\min}$. Compare the observed plots of these stationary oscillations with the plots of stationary (threshold) oscillations at exact tuning to resonance. What are the differences between the plots (and the phase trajectories) of stationary oscillations at the threshold within the interval of parametric excitation with friction, and the plots (and the phase trajectories) of stationary oscillations at the boundaries of the instability interval without friction?

(c) If the threshold is exceeded, why does the amplitude continue to increase indefinitely? In other words, why is friction unable to restrict the growth of the amplitude of parametrically excited oscillations?

(d) For small values of the modulation depth $m \ll 1$, calculate up to terms of second order in m the threshold value $Q = Q_{\min}$ of the quality factor for the period of modulation T lying somewhere within the interval of instability. Compare your theoretical result with the value which you have obtained experimentally in parts (a) and (b).

1.9*** **The Interval of Instability with Friction.**

(a) For some depth of modulation m, the frequency interval of parametric excitation shrinks because of friction and disappears as the quality factor reaches the threshold value. Let the quality factor Q be greater than the threshold value $Q_{\min}$. Find the values T_- and T_+ of the modulation period T which correspond to the boundaries of the instability interval for a given m and Q (in the approximation $m \ll 1$). Express these values in terms of m and $m_{\min}$, where $m_{\min} = \pi/(2Q)$ is the approximate threshold value of the modulation depth m for a given quality factor Q.

(b) In order to observe steady oscillations corresponding to these boundaries as soon as the simulation begins, you need to set the initial conditions properly. For a given value φ_0 of the initial deflection, and for each of the boundaries of the interval, what initial velocity produces steady oscillations from the very beginning? Verify your answer by simulating the experiment.

1.10 **Oscillations outside the Interval of Parametric Resonance.** For a given value of m, enter a value T of the modulation period lying somewhere outside the limits of the instability interval. Convince yourself that for any set of initial conditions the oscillations eventually fade away, even if the friction is very weak, and that the rotor comes to rest at the equilibrium position in spite of the forced periodic changes in its moment of inertia.

2.2 Parametric Resonances of High Orders

2.1* **The Third Parametric Resonance ($n = 3$) without Friction.**

(a) Examine the parametric excitation of the rotor for abrupt changes of its moment of inertia with the period $T \approx 3T_0/2$ (approximately one and a half times the natural period, or about three cycles of the parameter modulation during two cycles of natural oscillations). What initial conditions ensure the growth of the amplitude from the beginning of the modulation?

(b) What value m of the modulation depth in the absence of friction is necessary in order to double the initial oscillation during 15 cycles of the parameter modulation? After how many cycles does the amplitude double once more?

(c) For what initial conditions does the oscillation at first decrease? Why does this fading inevitably change after a while into an increase in the amplitude?

2.2* **The Threshold for the Third Resonance.**

(a) For small values m of the modulation depth, calculate the threshold value $Q_{\min}$ of the quality factor up to terms in the first order of m. How does this value depend on m? Compare your answer with the principal resonance, $n = 1$ (See Problem 1.3), and with the second resonance, $n = 2$ (See Problem 3.4). What might be a qualitative explanation for the difference?

(b) For $m \approx 30\%$ evaluate the minimal value $Q_{\min}$ of the quality factor for which parametric resonance of the order $n = 3$ is possible. Improve your theoretical estimate by simulating the experiment. Explain the observed shape of the angular velocity plot and the form of the phase trajectory of stationary oscillations at $Q = Q_{\min}$. What factor determines the amplitude of such oscillations?

2.3** **The Third Interval of Parametric Excitation.**

(a) Calculate the values of the modulation period T which, in the absence of friction, correspond to the boundaries of the third instability interval for a given modulation depth m (in the approximation $m \ll 1$). How does the width of the interval depend on the depth of modulation? Do the terms of the second order influence the width of the interval?

(b) What value of the initial angular velocity ought you to enter for a given initial deflection φ_0 in order to get stationary oscillations of a constant amplitude from the very beginning of the modulation on each boundary of the instability interval? Verify your calculated values experimentally.

What are the shapes of the phase trajectories that correspond to the left and right boundaries of this interval?

(c) Explore the width of the third interval of parametric excitation without friction at arbitrarily large values of the modulation depth m. Note how the interval moves to the left and gradually shrinks as m becomes greater. (See also figure 10.)

At m = 60% both boundaries of the interval coincide. (You may say also that they *intersect* at this value of m.) This coincidence means that for the corresponding value of the modulation period T you get steady oscillations for arbitrary initial conditions. What might be a physical explanation for this behavior?

Hint: What is the ratio of the natural periods for the maximal and minimal values of the moment of inertia for this value of the modulation depth?

2.4** **The Third Instability Interval with Friction.**

(a) At small values of m the third parametric resonance occurs at $T = 3T_0/2$. However, with the growth of m the resonant value of the modulation period T departs from $3T_0/2$. Find an analytical expression for this resonant value of T (for an arbitrarily large modulation depth m) and make a numerical estimate for m = 15% and m = 40%.

(b) How does friction reduce the width of the third interval of parametric excitation? For a small depth of modulation $m \ll 1$, calculate approximate values of the period of modulation T which correspond to the boundaries of the interval for a given value of the quality factor Q. Express the results in terms of m and the threshold value $m_{\min} = 3\pi/(2Q)$ (see Problem 2.2) for the given Q-value.

2.5** **Parametric Resonance of the Second Order** $(n = 2)$.

(a) Choosing a moderate value of the modulation depth ($m < 20\%$), excite parametric resonance of the order $n = 2$ (for which the period of modulation is approximately equal to the average natural period). Why does the growth in amplitude occur much more slowly in this case than it does for the principal resonance, and even than it does for the resonance of the order $n = 3$ (for the same value m of the modulation depth)? Explain the observed shape of the phase diagram for $n = 2$. Try to determine experimentally the threshold value of the modulation depth for a given value of the quality factor (say, $Q = 15$).

(b) For small values of the modulation depth $m \ll 1$, try to calculate the threshold value of the quality factor $Q_{\min}$. (You need to keep the terms of the second order in m). How does the threshold value of Q depend on m? Compare your calculated value with the threshold of principal resonance and of the third resonance. Explain the difference qualitatively. Compare also the theoretical threshold value with your experimental result of part (a).

2.6** **The Second Interval of Parametric Excitation.**

(a) For small values of the modulation depth $m \ll 1$, calculate the width of the interval (you need to keep the terms to the second order of m). How does the width depend on m?

(b) Excite and experimentally examine stationary oscillations without friction which correspond to the boundaries of the second instability interval (near the resonance for $n = 2$). For small values of the modulation depth $m \ll 1$, why is this interval considerably narrower than the interval for resonance of a higher order $n = 3$?

(c) Why do two different phase trajectories correspond to each boundary of the interval? What is the difference between the two stationary oscillations that correspond to the same boundary? How can each one of them be excited? What initial conditions ensure steady oscillations from the beginning of modulation?

2.7*** **The Second Interval of Parametric Excitation with Friction.**

How does friction influence the width of the second interval of parametric excitation? For a small depth of modulation $m \ll 1$, calculate approximate values of the period of modulation T which correspond to the boundaries of the interval for a given value of the quality factor. Write down the results in terms of m and the threshold value $Q_{\min}$ (see Problem 2.5) for the given value of Q.