The Rigid Pendulum – an Antique but Evergreen Physical Model

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Abstract

All kinds of motion of a rigid pendulum (including swinging with arbitrarily large amplitudes and complete revolutions) are investigated both analytically and with the help of computerized simulations based on the educational software package PHYSICS OF OSCILLATIONS developed by the author (see in the web http://www.aip.org/pas). The simulation experiments of the package reveal many interesting peculiarities of this famous physical model and aid greatly an understanding of basic principles of the pendulum motion. The computerized simulations complement the analytical study of the subject in a manner that is mutually reinforcing.

1 Introduction: The Investigated Physical System

The simple pendulum is a famous physical model frequently encountered in textbooks and papers primarily due to its important role in the history of physics. This versatile model is useful and interesting not only in itself as the most familiar example of a nonlinear mechanical oscillator but more importantly because many problems in various branches of physics can be reduced to the differential equation describing the motion of a pendulum. The theory of solitons (solitary wave disturbances traveling in nonlinear media with dispersion), the problem of superradiation in quantum optics, and Josephson effects in weak superconductivity are the most important examples.

One may wonder whether it is possible to add something new to the old problem of the pendulum motion. However, every generation of physics students, as well as their teachers, discover anew the attractiveness of this model and surprising beauty in various aspects of its behaviour.

In this paper we describe a combined analytical and computerized approach to the eternal problem of the pendulum motion. Our study is based on the usage of the educational software package PHYSICS OF OSCILLATIONS [1] developed by the author. This package of interactive programs is a kind of desk-top laboratory designed for exploration of the mathematical models of various linear and nonlinear mechanical oscillatory systems. The simulations allow us to directly observe the motion and to obtain graphs of the variables that describe the system along with phase diagrams and graphs of the energy transformations. The graphs and phase diagrams are plotted on the screen simultaneously with the display of the motion.

The simulations [1] bring to life many abstract concepts of the physics of oscillations. Varying experimental conditions, we can investigate interesting situations which are inaccessible in a real laboratory experiment. The screen displays subtle details that are easily missed in direct observation. With these highly interactive programs, the students have an opportunity to carry out interesting mini-research physics projects on their own.

One of the simulation programs in [1] is especially suited for exploration of an ordinary pendulum in a uniform gravitational field, that is, of any rigid body that can swing and rotate about some fixed horizontal axis (a *compound*, or *physical pendulum*). Its simplest form is represented by a massive small bob at the end of a rigid rod of negligible mass (a *simple pendulum*). We employ a rigid rod rather than a flexible string in order to examine complete revolutions of the pendulum as well as its swinging to and fro. Special attention in our discussion is devoted to cases in which the swing approaches 180°. Revolutions of the pendulum occurring when the total energy slightly exceeds the potential energy of the inverted pendulum are also investigated in detail.

In the state of stable equilibrium the center of mass of the pendulum is located vertically below the axis of rotation. When the pendulum is deflected from this position through an angle φ, the restoring torque of the gravitational force is proportional to $\sin\varphi$. In the case of small angles φ (i.e., for small oscillations of the pendulum) the values of the sine and of its argument nearly coincide ($\sin\varphi \approx \varphi$), and the pendulum behaves like a linear oscillator. In particular, in the absence of friction it executes *simple harmonic motion*. However, when the amplitude is large, the motion is oscillatory (and periodic in the absence of friction) but no longer simple harmonic. In this case, a graph of the angular displacement versus time noticeably departs from a sine curve, and the period of oscillation noticeably depends on the amplitude. Figure 1 shows the screen (illustrating the output of the corresponding program in [1]) with an example of time-dependent graphs of the angle and angular velocity for such non-sinusoidal oscillations whose amplitude equals 179.00 degrees. This screen shows also their spectrum, that is, the sinusoidal components (harmonics) with frequencies ω, 3ω, 5ω, etc. We note that the period $T = 2\pi/\omega$ of these oscillations (as well as the period of the principal harmonic component) equals almost four periods $T_0 = 2\pi/\omega_0$ of oscillations with infinitely small amplitudes.

If the angular velocity imparted to the pendulum at its initial excitation is great enough, the pendulum at first executes complete revolutions losing energy through friction, after which it oscillates to and fro.

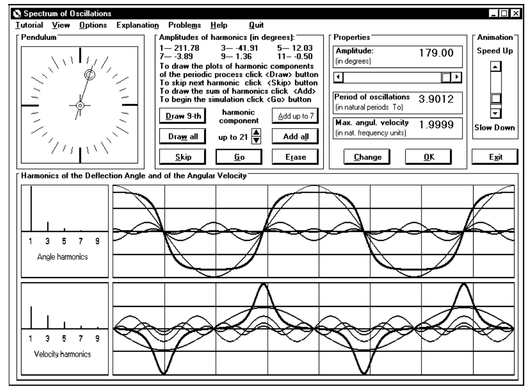

Figure 1: Spectrum of large oscillations.

2 The Physical Parameters and Differential Equation for the Pendulum

The equation of rotation of a solid about a fixed horizontal axis in the absence of friction in the case of a physical pendulum in a uniform gravitational field is:

$$J\ddot{\varphi} = -mga\sin\varphi. \tag{1}$$

Here J is the moment of inertia of the pendulum relative the axis of rotation, a is the distance between this axis and the center of mass, and g is the acceleration of gravity. The left-hand side of Eq. (1) is the time rate of change of the angular momentum, and the right-hand side is the restoring torque of the force of gravity. This torque is the product of the force mg (applied at the center of mass) and the lever arm $a\sin\varphi$ of this force. Dividing both sides of Eq. (1) by J we have:

$$\ddot{\varphi} + \omega_0^2\sin\varphi = 0, \tag{2}$$

where the notation $\omega_0^2 = mga/J$ is introduced.

For a simple pendulum $a = l$, $J = ml^2$, and so $\omega_0^2 = g/l$. For a physical pendulum, the expression for ω_0^2 can be written in the same form as for a simple pendulum provided we define a quantity l to be given by $l = J/(ma)$. It has the dimension of length, and is called the *reduced* or *effective* length of a physical pendulum. Since the differential equation of motion for a physical pendulum with an effective length l is the same as that for a simple pendulum of the same length l, the two systems are dynamically equivalent. This means that we need not distinguish one from the other in our investigation because both are described by the same mathematical model. At small angles of deflection from stable equilibrium, we can replace $\sin\varphi$ with φ in Eq. (2). Then Eq. (2) becomes the differential equation of motion of a linear oscillator, e.g., of a weighted spring that obeys Hook's law. Therefore, the quantity ω_0 in the differential equation of the pendulum, Eq. (2), has the physical sense of the angular frequency of small oscillations of the pendulum in the absence of friction. As an equivalent physical parameter of the pendulum, we can use the period of small non-damped oscillations $T_0 = 2\pi/\omega_0 = 2\pi\sqrt{l/g}$.

In the presence of a torque due to linear (say, viscous) friction, we must add a term to the right-hand side of Eq. (2) which is proportional to the angular velocity $\dot{\varphi}$. Thus, with friction included, the differential equation of the pendulum assumes the form:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2 \sin\varphi = 0. \tag{3}$$

Therefore the pendulum is characterized by two parameters: the *angular frequency* ω_0 of small free oscillations, and the *damping constant* γ, which has the dimensions of frequency (or of angular velocity). As in the case of a damped linear oscillator, it is convenient to use the dimentionless quality factor $Q = \omega_0/(2\gamma)$ rather than the damping constant γ to measure the effect of damping (see the following section).

The principal difference between Eq. (3) for the pendulum and the corresponding differential equation of motion for a linear oscillator (e.g., a weighted spring) is that Eq. (3) is a *nonlinear* differential equation. The difficulties in obtaining an analytical solution to Eq. (3) are caused by its nonlinearity. In the general case it is impossible to express the solution of Eq. (3) in elementary functions (although in the absence of friction the solution of Eq. (2) can be given in terms of special functions called elliptic integrals).

3 Viscous Damping of Small Oscillations

In the case of small amplitudes $\sin\varphi \approx \varphi$, and the exact equation of the pendulum, Eq. (3), can be approximated by the following linear equation:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2 \varphi = 0. \tag{4}$$

When friction is so weak that $\gamma < \omega_0$, the general solution of Eq. (4) has an oscillatory character and can be written in the form:

$$\varphi(t) = A_0 \exp(-\gamma t)\cos(\omega t + \delta_0). \tag{5}$$

In the case of weak or moderate damping, the time-dependent factor $A_0 \exp(-\gamma t)$ in (5) can be treated as an exponentially decreasing amplitude of diminishing oscillations. After an interval $\tau = 1/\gamma$ (after the *time of damping*), the amplitude is $e \approx 2.72$ times smaller than its initial value. The initial amplitude A_0 and the initial phase δ_0 depend on the mode of excitation. On the contrary, the frequency ω appearing in the cosine term in (5) is independent of the initial conditions being determined solely by the properties of the system:

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = \omega_0\sqrt{1 - (\gamma/\omega_0)^2}. \tag{6}$$

In the case of weak damping, when the constant γ is small compared to the frequency ω_0 of undamped oscillations, the frequency ω is very close to ω_0 (and the conventional period $T = 2\pi/\omega$ is very close to $T_0 = 2\pi/\omega_0$):

$$\omega \approx \omega_0 - \gamma^2/(2\omega_0), \quad T \approx T_0[1 + \gamma^2/(2\omega_0^2)].$$

The fractional difference of the frequencies $(\omega_0 - \omega_1)/\omega_0$ (and the periods) is proportional to the square of the small parameter γ/ω_0.

When $\gamma \ll \omega_0$, or $\tau \gg T_0 = 2\pi/\omega_0$ (condition of weak damping), the oscillator executes a large number N of oscillations during the decay time τ:

$N = \tau/T_0 \gg 1$. Consecutive maximal deflections from the equilibrium position diminish in a geometric progression. Letting φ_n be the maximal angular displacement of the n-th oscillation, we have

$$\varphi_{n+1}/\varphi_n \approx \exp(-\gamma T_0) \approx 1 - \gamma T_0.$$

That is, the ratio of successive terms in this infinite geometric progression is less than unity by the small value $\gamma T_0 = T_0/\tau \ll 1$.

The strength of viscous friction in the system is usually characterized either by the damping constant γ, which, as can be seen from Eq. (4), has the dimension of frequency, or by a more convenient dimensionless quantity Q, called the *quality factor.* The quality factor Q is defined as multiplied by 2π ratio of the energy stored by the oscillator to the energy dissipated through friction during a period. It follows from this definition that Q equals the ratio of ω_0 to 2γ (see, for example, [2]):

$$Q = \frac{\omega_0}{2\gamma} = \pi\frac{\tau}{T_0}. \tag{7}$$

The number of cycles during which the amplitude of oscillations decreases by a factor $e \approx 2.72$ is given by Q/π, and the number of cycles $N_{1/2}$ during which the amplitude is halved is given by:

$$N_{1/2} = (\ln 2/\pi)Q = 0.22\,Q = Q/4.53. \tag{8}$$

A graph of damped oscillations is shown in Figure 2. For the value $Q = 18.1$ the amplitude halves after four cycles. It halves again (up to 1/4 of its initial value) after the next four cycles of oscillations.

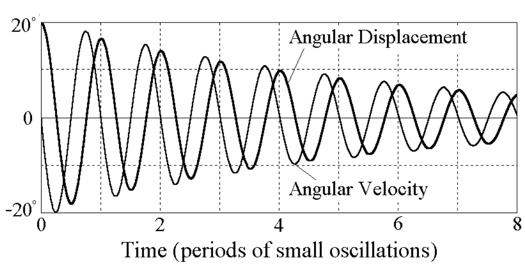

Figure 2: Graphs of damped oscillations.

Free oscillations are characterized by the *exchange of energy* between its potential and kinetic forms. At the points of maximum displacement from the equilibrium position, the kinetic energy is zero, and the total energy is the potential energy. A quarter of a period later the pendulum passes through the equilibrium position where the potential energy is zero, and the total energy is the kinetic energy. During the next quarter of a period, the reverse exchange of energy occurs. Such transformations happen twice during one period. That is, oscillations of the two kinds of energy occur 180° out of phase with one another with double the natural frequency of the pendulum. Figure 3 shows the transformations of energy occurring in damped oscillations, along with the plots of the angular displacement and the angular velocity (for $Q = 9.1$).

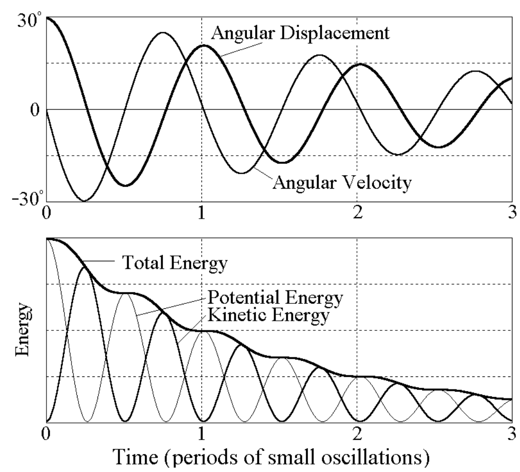

Figure 3: Graphs of the energy transformations during damped natural oscillations.

In the presence of friction, the exchanges between kinetic and potential energy are partially irreversible because of the dissipation of energy. This dissipation occurs nonuniformly during a complete cycle: its rate, $-dE/dt$, is zero when the pendulum, in a given cycle, is at the extremes of its motion and its angular velocity, $\dot{\varphi}$, is zero. The rate of dissipation is greatest when the pendulum is moving in the vicinity of the equilibrium position, and its angular velocity is maximal. Indeed, the rate of energy dissipation $-dE/dt = -N_{\rm fr}\dot\varphi$ caused by viscous friction is proportional to the square of the angular velocity and hence to the momentary value of the kinetic energy $J\dot\varphi^2/2$ of the pendulum. The statement, frequently encountered in textbooks, that the energy of a damped oscillator decays exponentially, is valid only on the average: It applies strictly only to values of the total energy at discrete time instants separated by a period (and half-period).

4 The Period of Small Oscillations

Nonlinear character of the pendulum is revealed primarily in dependence of the period of oscillations on the amplitude. To find an approximate formula for this dependence, we should keep the next term in the expansion of $\sin\varphi$ in Eq. (2) into the power series:

$$\sin\varphi \approx \varphi - \frac{1}{6}\varphi^3. \tag{9}$$

An approximate solution to the corresponding nonlinear differential equation (for the conservative pendulum with $\gamma = 0$),

$$\ddot\varphi + \omega_0^2\varphi - \frac{1}{6}\omega_0^2\varphi^3 = 0, \tag{10}$$

can be searched as a superposition of the sinusoidal oscillation $\varphi(t) = \varphi_{\rm m}\cos\omega t$ and its third harmonic $\epsilon\varphi_{\rm m}\cos 3\omega t$ whose frequency equals 3ω. (We assume $t = 0$ to be the moment of maximal deflection). This solution is found in many textbooks (see, for example, [3]). The corresponding derivation is a good exercise for students, allowing them to get an idea of analytical perturbational methods. The fractional contribution ϵ of the third harmonic equals $\varphi_{\rm m}^2/192$, where $\varphi_{\rm m}$ is the amplitude of the principal harmonic component whose frequency ω differs from the limiting frequency ω_0 of small oscillations by a term proportional to the square of the amplitude:

$$\omega \approx \omega_0(1 - \varphi_{\rm m}^2/16), \quad T \approx T_0(1 + \varphi_{\rm m}^2/16). \tag{11}$$

The same approximate formula for the period, Eq. (11), can be obtained from the exact solution expressed in terms of elliptic integrals (see, for example, the textbooks [3], [4], or [5]) by expanding the exact solution into a power series with respect to the amplitude $\varphi_{\rm m}$.

Equation (11) shows that, say, for $\varphi_{\rm m} = 30^\circ$ (0.52 rad) the fractional increment of the period (compared to the period of infinitely small oscillations) equals 0.017 (1.7%). The fractional contribution of the third harmonic in this non-sinusoidal oscillation equals 0.14%, that is, its amplitude equals only 0.043°.

The simulation program [1] allows us to verify this approximate formula for the period. The table below gives the values of T (for several values of the amplitude) calculated with the help of Eq. (11) and measured in the computational experiment. Comparing the values in the last two columns, we see that the approximate formula, Eq. (11), gives the value of the period for the amplitude of 45° with an error of only $(1.0400 - 1.0386)/1.04 = 0.0013 = 0.13\%$. However, for 90° the error is already 2.24%. The error does not exceed 1% for amplitudes up to 70°.

Amplitude $\varphi_{\rm m}$		T/T_0 (calculated)	T/T_0 (measured)
30°	($\pi/6$)	1.0171	1.0175
45°	($\pi/4$)	1.0386	1.0400
60°	($\pi/3$)	1.0685	1.0732
90°	($\pi/2$)	1.1539	1.1803
120°	($2\pi/3$)	1.2742	1.3730
135°	($3\pi/4$)	1.3470	1.5279
150°	($5\pi/6$)	1.4284	1.7622

5 The Phase Portrait and Large Oscillations

The mechanical state of a pendulum at any instant is determined by the two quantities: the *angular displacement* φ and the *angular velocity* $\dot\varphi$ (or instead by the *angular momentum* $J\dot\varphi$). The evolution of the mechanical state of the system during its entire motion can be graphically demonstrated very clearly by a *phase diagram,* i.e., a graph which plots the angular velocity $\dot{\varphi}$ (or the angular momentum $J\dot{\varphi}$) versus the angular displacement φ. In general, the structure of a phase diagram tells us a great deal about the possible motions of a nonlinear physical system.

If the motion of the physical system is *periodic*, the representative point, moving clockwise, generates a *closed path* in the phase plane. The phase trajectory of a periodic motion is closed, because the system returns to the same mechanical state after a full cycle.

The phase diagram for harmonic oscillations (e.g., for oscillations in a linear system without friction) is an ellipse (or a circle at the appropriate choice of the scales). The points of intersection of the phase curve with the φ-axis correspond to maximal deflections of the pendulum from the equilibrium position. At these turning points, the sign of the angular velocity $\dot{\varphi}$ changes, and the tangent to the phase curve is perpendicular to the abscissa axis.

We can construct the family of phase trajectories for a conservative system (e.g., for the pendulum) without explicitly solving the differential equation of motion of the system. The equations for phase trajectories follow directly from the law of the conservation of energy. The potential energy $E_{\mathrm{pot}}(\varphi)$ of a pendulum in the gravitational field depends on the angle of deflection φ measured from the equilibrium position:

$$E_{\mathrm{pot}}(\varphi) = mga(1 - \cos\varphi). \tag{12}$$

A graph of $E_{\mathrm{pot}}(\varphi)$ is shown in the upper part of Figure 4. The potential energy of the pendulum has a minimal value of zero in the lower stable equilibrium position (at $\varphi = 0$), and a maximal value of $2mga$ in the inverted position (at $\varphi = \pm\pi$) of unstable equilibrium.
(The maximal value of the potential energy is assumed to be the unit of energy in Figure 4). The dashed line shows the parabolic potential well for a linear oscillator whose period is independent of the amplitude (and of the energy) and equals the period of infinitely small oscillations of the pendulum.

In the absence of friction, the total energy E of the pendulum, i.e., the sum of its kinetic energy, $\frac{1}{2}J\dot{\varphi}^2$, and potential energy, remains constant during the motion:

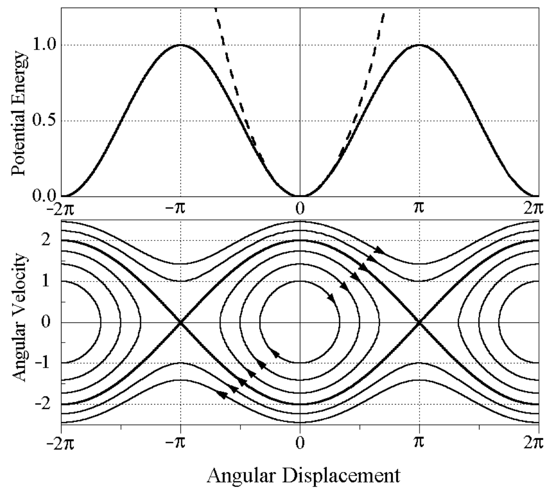

Figure 4: The potential well and the phase portrait of the conservative pendulum.

$$\frac{1}{2}J\dot{\varphi}^2 + mga(1 - \cos\varphi) = E. \tag{13}$$

This equation gives the relation between $\dot{\varphi}$ and φ, and therefore is the equation of the phase trajectory which corresponds to a definite value E of total energy. It is convenient to express Eq. (13) in a slightly different form. Recalling that $mga/J = \omega_0^2$ and defining the quantity $E_0 = J\omega_0^2/2$ (the quantity E_0 has the physical sense of the kinetic energy of a body with the moment of inertia J, rotating with the angular velocity ω_0), we rewrite Eq. (13):

$$\frac{\dot{\varphi}^2}{\omega_0^2} + 2(1 - \cos\varphi) = \frac{E}{E_0}. \tag{14}$$

The phase trajectory of a conservative system is symmetric about the horizontal φ-axis. This symmetry means that the motion of the pendulum in the clockwise direction is mechanically the same as the motion in the counterclockwise direction. In other words, the motion of a conservative system is *reversible*: if we instantaneously change the sign of its velocity, the representative point jumps to the symmetric position of the same phase trajectory on the other side of the horizontal φ-axis. In the reverse motion the pendulum passes through each spatial point φ with the same speed as in the direct motion. Since changing the sign of the velocity ($\dot{\varphi} \rightarrow -\dot{\varphi}$) is the same as changing the sign of time ($t \rightarrow -t$), this property of a conservative system is also referred to as the *symmetry of time reversal.*

The additional symmetry of the phase trajectories of the conservative pendulum about the vertical $\dot{\varphi}$-axis (with respect to the change $\varphi \rightarrow -\varphi$) follows from the symmetry of its potential well: $E_{\mathrm{pot}}(-\varphi) = E_{\mathrm{pot}}(\varphi)$.

If the total energy E of the pendulum is less than the maximal value of its potential energy ($E < 2mga = 4E_0$), that is, if the total energy is less than the height of the potential barrier in Figure 4, the pendulum swings to and fro between the extreme deflections φ_{m} and $-\varphi_{\mathrm{m}}$. These angles correspond to the extreme points at which the potential energy $E_{\mathrm{pot}}(\varphi)$ becomes equal to the total energy E of the pendulum. If the amplitude is small ($\varphi_{\mathrm{m}} \ll \pi/2$), the oscillations are nearly sinusoidal in time, and the corresponding phase trajectory is nearly an ellipse. The elliptical shape of the curve follows from Eq. (14) if we substitute there the approximate expression $\cos\varphi \approx 1 - \varphi^2/2$ valid for small angles φ:

$$\frac{\dot{\varphi}^2}{E\omega_0^2/E_0} + \frac{\varphi^2}{E/E_0} = 1. \tag{15}$$

This is the equation of an ellipse in the phase plane (φ, $\dot{\varphi}$). Its horizontal semiaxis equals the maximal deflection angle $\varphi_{\mathrm{m}} = \sqrt{E/E_0}$. Since both semiaxes are proportional to $\sqrt{E}$, the ellipses for different energies E (while $E \ll E_0$) are homothetic. If the angular velocity $\dot{\varphi}$ on the ordinate axis is plotted in units of the angular frequency ω_0 of small free oscillations, the ellipse (15) becomes a circle.

The shape of the closed phase trajectory, elliptical at small amplitudes, gradually changes as the amplitude and the energy are increased. The width of the phase trajectory (along φ-axis) increases more rapidly than does its height as the total energy E increases to $2mga$. The phase trajectory is stretched horizontally because for the same total energy the amplitude of oscillations in the potential well of the pendulum is greater than it is in the parabolic potential well of the linear oscillator. The greater the total energy E (and thus the greater the amplitude φ_{m}), the greater the departure of the phase trajectory from an ellipse and the greater the departure of the motion from simple harmonic.

With the growth of the angular displacement the restoring torque for the pendulum increases not as rapidly, and the slopes of the potential well rise not as steeply as for the linear oscillator: The pendulum is a system with a "soft" restoring torque. Therefore the period of oscillations, while independent of the amplitude for the linear oscillator, grows with the amplitude for the pendulum.

When the pendulum is deflected from the vertical position by an angle in the neighborhood of $90°$, the restoring torque of the gravitational force is almost constant: its dependence on φ is insignificant while φ varies in the vicinity of $90°$. The slope of the graph of potential energy versus φ is nearly constant in the vicinity of its point of inflection $\varphi = 90°$. Therefore the pendulum moves there with almost constant angular acceleration, and the time dependence of its angular velocity is nearly linear. Hence at large amplitudes ($110 - 140°$) the graph of the angular velocity is almost saw-toothed. For motion with constant acceleration, the graph of displacement is a parabola. Therefore the segments of the graph of $\varphi(t)$ (which correspond to rectilinear segments of the $\dot{\varphi}(t)$ graph) are nearly parabolic.

At large amplitudes the pendulum spends more time near the extreme points (or *turning points*) at which its direction of motion is reversed. The crests of the graph of $\varphi(t)$ are flattened, and those of the $\dot{\varphi}(t)$ graph are sharpened. These changes in the shape of the graphs and in the period of oscillations are clearly seen in Figure 5.

The lower part of Figure 5 shows time-dependent graphs of the potential, kinetic, and total energies of the pendulum. In the case of a linear oscillator whose potential well is parabolic, time dependencies of both potential and kinetic energies are sinusoidal, and their time average values are equal to one another. The upper slopes of the potential well of the pendulum are not as steep as those of the parabola, and so the pendulum spends more time near the ex-

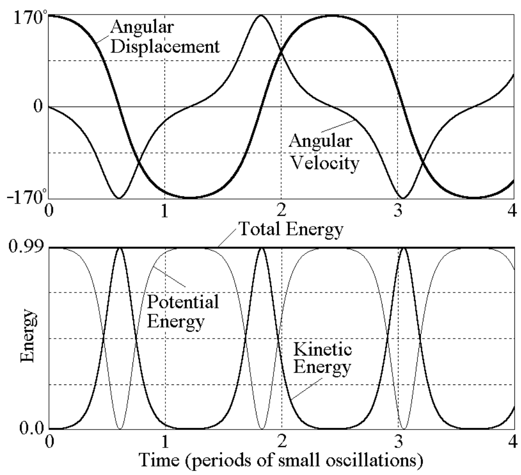

Figure 5: The graphs of large oscillations in the absence of friction ($\varphi_{\max} = 170°$).

treme points at large deflections, where its potential energy is greater than kinetic energy. Crests of the graph of the potential energy $E_{\text{pot}}(t)$ become wider than the valleys between them. The opposite changes occur with the graph of the kinetic energy $E_{\text{kin}}(t)$. Although maximum values of both potential and kinetic energies are equal to the constant value of total energy E, their time-averaged values become different as the amplitude is increased: The average value of the potential energy becomes greater than that of the kinetic energy.

If $E > 2mga$, the kinetic energy and the angular velocity of the pendulum are non-zero even at $\varphi = \pm\pi$. In contrast to the case of swinging, now the angular velocity does not change its sign. The pendulum executes *rotation* in a full circle. This rotation is nonuniform. When the pendulum passes through the lowest point, its angular velocity is greatest in magnitude.

In the phase plane, rotation of the pendulum is represented by the paths which continue beyond the vertical lines $\varphi = \pm\pi$, repeating themselves every full cycle of revolution, as shown in Figure 4. Upper paths lying above the φ-axis, where $\dot{\varphi}$ is positive and φ grows in value, correspond to counterclockwise rotation, and paths below the axis, along which the representative point moves from the right to the left, correspond to clockwise rotation of the pendulum.

The angles φ and $\varphi \pm 2\pi$, $\varphi \pm 4\pi$, ... denote the same position of the pendulum and thus are equivalent. Thus it is sufficient to consider only a part of the phase plane, e.g., the part enclosed between the vertical lines $\varphi = -\pi$ and $\varphi = \pi$ (see Figure 4). The cyclic motion of the pendulum in the phase plane is then restricted to the region lying between these vertical lines. We can identify these lines and assume that when the representative point leaves the region crossing the right boundary $\varphi = \pi$, it enters simultaneously from the opposite side at the left boundary $\varphi = -\pi$ (for a counterclockwise rotation of the pendulum).

We can imagine the two-dimensional phase space of a rigid pendulum not only as a part of the plane $(\varphi, \dot{\varphi})$ enclosed between the vertical lines $\varphi = +\pi$ and $\varphi = -\pi$, but also as a continuous surface, namely, the surface of a cylinder. We may do so because opposing points on these vertical lines have the same value of $\dot{\varphi}$ and describe physically equivalent mechanical states, and the dependence of the restoring gravitational torque on φ is periodic. (The potential energy $E_{\text{pot}}(\varphi) = mga(1 - \cos\varphi)$ is periodic). Therefore we can cut out this part of the phase plane and roll it into a cylinder so that the bounding lines $\varphi = +\pi$ and $\varphi = -\pi$ are joined. A phase curve circling around the cylinder corresponds to a nonuniform rotational motion of the pendulum.

6 Limiting Motion along the Separatrix

The phase trajectory corresponding to a total energy E which is equal to the maximal possible potential energy, namely $E_{\text{pot}}(\pi) = 2mga$, is of special interest. It separates the central region of the phase plane which is occupied by the closed phase trajectories of oscillations from the outer region, occupied by the phase trajectories of rotations. This boundary is called the *separatrix*. In Figure 4 it is singled out by a thick line. The separatrix divides the phase plane of a conservative pendulum into regions which correspond to different types of motion. The equation of the separatrix follows from Eq. (13) by setting $E = 2mga$, or from Eq. (14) by setting $E = 4E_0 = 2J\omega_0^2$:

$$\dot{\varphi} = \pm 2\omega_0 \cos(\varphi/2). \qquad (16)$$

The limiting motion of a conservative pendulum with total energy $E = 2mga$ is worth a detailed investigation. In this case the representative point in the phase plane moves along the separatrix.

When the pendulum with the energy $E = 2mga$ approaches the inverted position at $\varphi = \pi$ or $\varphi = -\pi$, its velocity approaches zero, becoming zero at $\varphi = \pm\pi$. This state is represented in the phase plane by the saddle points $\varphi = \pi$, $\dot{\varphi} = 0$ and $\varphi = -\pi$, $\dot{\varphi} = 0$ where the upper and lower branches of the separatrix (Eq. (16)) meet on the φ-axis. Both these points represent the same mechanical state of the system, that in which the pendulum is at rest in the unstable inverted position. The slightest initial displacement of the pendulum from this point to one side or the other results in its swinging with an amplitude which almost equals π, and the slightest initial push causes rotational motion of the pendulum in a full circle. With such swinging or rotation, the pendulum remains in the vicinity of the inverted position for an extended time.

For the case of motion along the separatrix, i.e., for the motion of the pendulum with total energy $E = 2mga = 4E_0$, there exist an analytical solution (in elementary functions) for the angle of deflection $\varphi(t)$ and for the angular velocity $\dot{\varphi}(t)$. Integration of the differential equation Eq. (16) with respect to time (for the positive sign of the root) at the initial condition $\varphi(0) = 0$ yields:

$$-\omega_0 t = \ln\, \tan[(\pi - \varphi)/4], \qquad (17)$$

whence we obtain for $\varphi(t)$:

$$\varphi(t) = \pi - 4\arctan(e^{-\omega_0 t}). \qquad (18)$$

This solution describes a counterclockwise motion beginning at $t = -\infty$ from $\varphi = -\pi$. At $t = 0$ the pendulum passes through the bottom of its circular path, and continues its motion until $t = +\infty$, asymptotically approaching $\varphi = +\pi$. A graph of $\varphi(t)$ for this motion is shown in Figure 6.

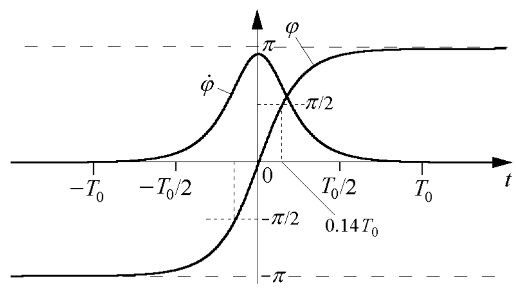

Figure 6: The graphs of φ and $\dot{\varphi}$ for the limiting motion (total energy $E = 2mga = 4E_0$).

The second solution which corresponds to the clockwise motion (to the motion along the other branch of the separatrix in the phase plane) can be obtained from Eq. (18) by the transformation of time reversal. Solutions with different initial conditions can be obtained from Eq. (18) simply by a shift of the time origin (by the substitution of $t - t_0$ for t).

To obtain the time dependence of the angular velocity $\dot{\varphi}(t)$ for the limiting motion of the pendulum, we can express $\cos(\varphi/2)$ from Eq. (18) as a function of time t:

$$\cos(\varphi/2) = \frac{1}{\cosh(\omega_0 t)}.$$

After substitution of this value into Eq. (16), we obtain the time dependence of $\dot{\varphi}$:

$$\dot{\varphi}(t) = \pm\frac{2\omega_0}{\cosh(\omega_0 t)} = \pm\frac{4\omega_0}{(e^{\omega_0 t} + e^{-\omega_0 t})}. \qquad (19)$$

A graph of $\dot{\varphi}(t)$ is also shown in Figure 6. The graph of this function has the form of an isolated impulse. In Eq. (19) the origin $t = 0$ is chosen to be the instant at which the pendulum passes through the equilibrium position with the angular velocity $\dot{\varphi} = \pm 2\omega_0$. This moment corresponds to the peak value of the impulse. For time $t = \pm T_0/2$ on either side of the peak Eq. (19) gives the angular velocity of only

$\pm 0.17\,\omega_0$. Therefore the period $T_0 = 2\pi/\omega_0$ of small oscillations can be assumed to be an estimate for the duration of the impulse on the velocity graph, that is, for the time needed for the pendulum to execute almost all of its circular path, from the vicinity of the inverted position through the lowest point and back.

Using the analytical expression for the time dependence of the angular velocity given by Eq. (19), we can calculate the time interval during which the pendulum moves from one horizontal position to the other, passing through the lower equilibrium position. During this time the kinetic energy of the pendulum is greater than its potential energy. For the time dependence of the kinetic energy during the limiting motion we obtain:

$$E_{\rm kin} = \frac{1}{2}J\dot{\varphi}^2 = \frac{8J\omega_0^2}{(e^{\omega_0 t} + e^{-\omega_0 t})^2}. \qquad (20)$$

To find the instants that correspond to horizontal positions of the pendulum we equate $E_{\rm kin}(t)$ from Eq. (20) to the potential energy of the pendulum in the horizontal position $mga = J\omega_0^2$. Thus for the interval τ between these instants we obtain the value $0.28\,T_0$.

The wings of the profile decrease exponentially as $t \to \pm\infty$. Actually, for large positive values of t, we may neglect the second term $\exp(-\omega_0 t)$ in the denominator of Eq. (19), and we find that:

$$\dot{\varphi}(t) \approx \pm 4\omega_0 e^{-\omega_0 t}. \qquad (21)$$

Thus, in the limiting motion of the representative point along the separatrix, when the total energy E is exactly equal to the height $2mga$ of the potential barrier, the speed of the pendulum decreases steadily as it nears the inverted position of unstable equilibrium. The pendulum approaches the inverted position asymptotically, requiring an infinite time to reach it. The motion is not periodic.

The mathematical relationships associated with the limiting motion of a pendulum along the separatrix play an important role in the theory of solitons (see [6]).

7 Period of Large Oscillations and Revolutions

If the energy differs from the critical value $2mga$, the motion of the pendulum in the absence of friction (swinging at $E < 2mga$ or rotation at $E > 2mga$) is periodic. The period T of such a motion the greater the closer the energy E to $2mga$. Figure 7 gives the dependence of the period on the total energy $T(E)$ obtained with the help of the simulation program [1]. (The energy is measured in units of the maximal potential energy $2mga$.)

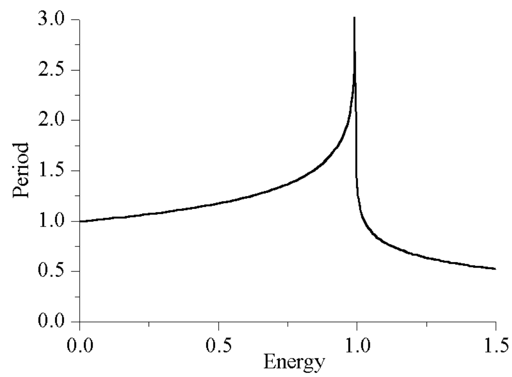

Figure 7: The period versus total energy.

The initial almost linear growth of the period with E corresponds to the approximate formula, Eq. (11). Indeed, Eq. (11) predicts a linear dependence of T on $\varphi_{\rm m}^2$, and for small amplitudes $\varphi_{\rm m}$ the energy is proportional to the square of the amplitude. When the energy approaches the value $2mga$, the period grows infinitely. Greater values of the energy correspond to the rotating pendulum. The period of rotation decreases with the energy. The asymptotic behavior of the period at $E \gg 2mga$ can be found as follows.

When the total energy E of the pendulum is considerably greater than the maximal value $2mga$ of its potential energy, we can assume all the energy of the pendulum to be the kinetic energy of its rotation. In other words, we can neglect the influence of the gravitational field on the rotation and consider this rotation to be uniform. The angular velocity of this rotation is approximately equal to the angular velocity Ω received by the pendulum at the initial excitation. The period T of rotation is inversely proportional to the angular velocity of rotation: $T = 2\pi/\Omega$. So for $E = J\Omega^2/2 \gg 2mga$ the asymptotic dependence of the period on the initial angular velocity is the inverse proportion: $T \propto 1/\Omega$.

To find the dependence $T(\Omega)$ more precisely, we need to take into account the variations in the angular velocity caused by gravitation. The angular velocity of the pendulum oscillates between the maximal value Ω in the lower position and the minimal value $\Omega_{\min}$ in the upper position. The latter can be found from the conservation of energy (see [7] for details):

$$\Omega_{\min} = \sqrt{\Omega^2 - 4\omega_0^2} \approx \Omega\left(1 - 2\frac{\omega_0^2}{\Omega^2}\right).$$

For rapid rotation we can assume these oscillations of the angular velocity to be almost sinusoidal. Then the average angular velocity of rotation is approximately the half-sum of its maximal and minimal values:

$$\Omega_{\text{av}} \approx (\Omega + \Omega_{\min})/2 = \Omega(1 - \omega_0^2/\Omega^2),$$

and the period of rotation is:

$$T(\Omega) = \frac{2\pi}{\Omega_{\text{av}}} \approx T_0\,\frac{\omega_0}{\Omega}\left(1 + \frac{\omega_0^2}{\Omega^2}\right).$$

However, the most interesting peculiarities are revealed if we investigate the dependence of the period on energy in the vicinity of $E_{\max} = 2mga$.

Measuring the period of oscillations successively for the amplitudes 179.900°, 179.990°, and 179.999° (each with an initial angular velocity of zero), we see that duration of the impulses on the graph of the angular velocity very nearly remains the same, but the intervals between them become longer as the amplitude approaches 180°: Experimental values of the period T of such extraordinary oscillations are respectively $5.5\,T_0$, $6.8\,T_0$, and $8.3\,T_0$.

It is interesting to compare the motions for two values of the total energy E which differ slightly from $E_{\max}$ on either side by the same amount, i.e., for $E/E_{\max} = 0.9999$ and $E/E_{\max} = 1.0001$. In the phase plane, these motions occur very near to the separatrix, the first one inside and the latter outside of the separatrix. The inner closed curve corresponds to oscillations with the amplitude 178.9°. Measuring the periods of these motions, we obtain the values $3.814\,T_0$ and $1.907\,T_0$ respectively. That is, the measured period of these oscillations is exactly twice the period of rotation, whose phase curve adjoins the separatrix from the outside.

The graphs of $\varphi(t)$ and $\dot\varphi(t)$ for oscillations and revolutions of the pendulum whose energy equals $E_{\max} \mp \Delta E$ are shown respectively in the upper and lower parts of Figure 8.

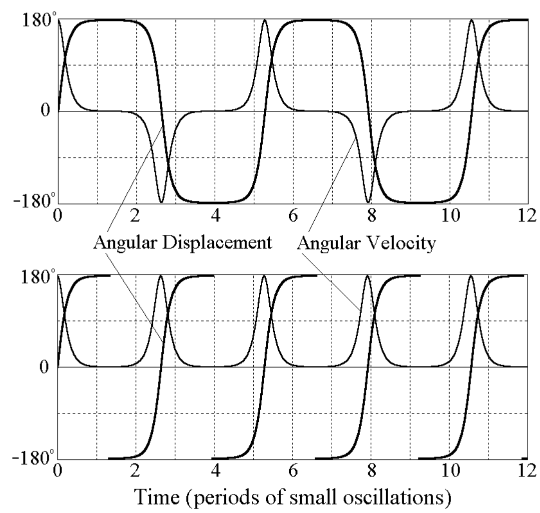

Figure 8: The graphs of $\varphi(t)$ and $\dot\varphi(t)$ for the pendulum excited at $\varphi = 0$ by imparting the initial angular velocity of $\dot\varphi = 2\omega_0(1 \mp 10^{-6})$.

Next we suggest a theoretical approach which can be used to calculate the period of oscillations and revolutions with $E \approx 2mga$.

From the simulation experiments we can conclude that during the semicircular path, from the equilibrium position up to the extreme deflection or to the inverted position, both of the motions shown in Figure 8 almost coincide with the limiting motion (Figure 6). These motions differ from the limiting motion appreciably only in the immediate vicinity of the extreme point or near the inverted position: In the first case the pendulum stops at this extreme point and then begins to move backwards, while in the limiting motion the pendulum continues moving for an unlimited time towards the inverted position; in the second case the pendulum reaches the inverted position during a finite time.

For the oscillatory motion under consideration, the representative point in the phase plane generates a closed path during one cycle, passing along both branches of the separatrix. In this motion the pendulum goes twice around almost the whole circle, covering it in both directions. On the other hand, executing rotation, the pendulum makes one circle during a cycle of revolutions, and the representative point passes along one branch of the separatrix (upper or lower, depending on the direction of rotation). To explain why the period of these oscillations is twice the period of corresponding revolutions, we must show that the motion of the pendulum with energy $E = 2mga - \Delta E$ from $\varphi = 0$ up to the extreme point requires the same time as the motion with the energy $E = 2mga + \Delta E$ from $\varphi = 0$ up to the inverted vertical position.

Almost all of both motions occurs very nearly along the same path in the phase plane, namely, along the separatrix from the initial point $\varphi = 0$, $\dot{\varphi} \approx 2\omega_0$ up to some angle φ_0 whose value is close to π. To calculate the time interval required for this part of the motion, we can assume that the motion (in both cases) occurs exactly along the separatrix, and take advantage of the corresponding analytical solution, expressed by Eq. (18).

Assuming $\varphi(t)$ in Eq. (18) to be equal to φ_0, we can find the time t_0 during which the pendulum moves from the equilibrium position $\varphi = 0$ up to the angle φ_0 (for both cases):

$$\omega_0 t_0 = -\ln\tan\frac{\pi - \varphi_0}{4} = -\ln\tan\frac{\alpha_0}{4}, \qquad (22)$$

where we have introduced the notation $\alpha_0 = \pi - \varphi_0$ for the angle that the rod of the pendulum at $\varphi = \varphi_0$ forms with the upward vertical line. When φ_0 is close to π, the angle α_0 is small, so that in Eq. (22) we can assume $\tan(\alpha_0/4) \approx \alpha_0/4$, and $\omega_0 t_0 \approx \ln(4/\alpha_0)$.

Later we shall consider in detail the subsequent part of motion which occurs from this arbitrarily chosen angle $\varphi = \varphi_0$ towards the inverted position, and prove that the time t_1 required for the pendulum with the energy $2mga + \Delta E$ (rotational motion) to reach the inverted position $\varphi = \pi$ equals the time t_2 during which the pendulum with the energy $2mga - \Delta E$ (oscillatory motion) moves from φ_0 up to its extreme deflection φ_{m}, where the angular velocity becomes zero, and the pendulum begins to move backwards.

When considering the motion of the pendulum in the vicinity of the inverted position, we find it convenient to define its position (instead of the angle φ) by the angle α of deflection from this position of unstable equilibrium. This angle equals $\pi - \varphi$, and the angular velocity $\dot{\alpha}$ equals $-\dot{\varphi}$. The potential energy of the pendulum (measured relative to the lower equilibrium position) depends on α in the following way:

$$E_{\mathrm{pot}}(\alpha) = mga(1 + \cos\alpha) \approx 2mga(1 - \frac{1}{4}\alpha^2). \qquad (23)$$

The latter expression is valid only for relatively small values of α, when the pendulum moves near the inverted position. Phase trajectories of motion with energies $E = 2mga \pm \Delta E$ near the saddle point $\varphi = \pi$, $\dot{\varphi} = 0$ (in the new variables $\alpha = 0, \dot{\alpha} = 0$) can be found from the conservation of energy using the approximate expression (23) for the potential energy:

$$\frac{1}{2}J\dot{\alpha}^2 + \frac{1}{2}mga\alpha^2 = \pm\Delta E, \text{ or } \frac{\dot{\alpha}^2}{\omega_0^2} - \alpha^2 = \pm 4\varepsilon. \qquad (24)$$

Here we use the notation $\varepsilon = \Delta E/E_{\max} = \Delta E/(2mga)$ for the small ($\varepsilon \ll 1$) dimensionless quantity characterizing the fractional deviation of energy E from its value $E_{\max}$ for the separatrix. It follows from Eq. (24) that phase trajectories near the saddle point are hyperbolas whose asymptotas are the two branches of the separatrix that meet at the saddle point. Part of the phase portrait near the saddle point is shown in Figure 9. The curve *1* for the energy $E = 2mga + \Delta E$ corresponds to the rotation of the pendulum. It intersects the ordinate axis when the pendulum passes through the inverted position. The curve *2* for the energy $E = 2mga - \Delta E$ describes the oscillatory motion. It intersects the abscissa axis at the distance $\alpha_{\rm m} = \pi - \varphi_{\rm m}$ from the origin. This is the point of extreme deflection in the oscillations.

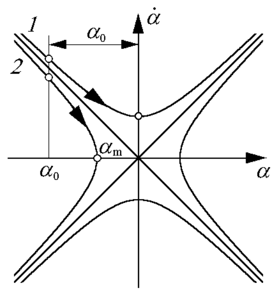

Figure 9: Phase curves near the saddle point.

For $\alpha \ll 1$ the torque of the gravitational force is approximately proportional to the angle α, but in contrast to the case of stable equilibrium, the torque $N = -dE_{\rm pot}(\alpha)/d\alpha = mga\alpha$ tends to move the pendulum farther from the position $\alpha = 0$ of unstable equilibrium. Substituting the torque N in the law of rotation of a solid, we find the differential equation of the pendulum valid for its motion near the point $\alpha = 0$:

$$J\,\ddot\alpha = mga\,\alpha, \quad \text{or} \quad \ddot\alpha - \omega_0^2\alpha = 0. \tag{25}$$

The general solution of this linear equation can be represented as a superposition of two exponential functions of time t:

$$\alpha(t) = C_1 e^{\omega_0 t} + C_2 e^{-\omega_0 t}. \tag{26}$$

Next we consider separately the two cases of motion with the energies $E = E_{\rm max} \pm \Delta E$.

1. Rotational motion ($E = E_{\rm max} + \Delta E$) along the curve *1* from α_0 up to the intersection with the ordinate axis. Let $t = 0$ be the moment of crossing the inverted vertical position: $\alpha(0) = 0$. Hence in Eq. (26) $C_2 = -C_1$. Then from Eq. (24) $\dot\alpha(0) = 2\sqrt{\varepsilon}\omega_0$, and $C_1 = \sqrt{\varepsilon}$. To determine duration t_1 of the motion, we assume in Eq. (26) $\alpha(t_1) = \alpha_0$:

$$\alpha_0 = \sqrt{\varepsilon}(e^{\omega_0 t_1} - e^{-\omega_0 t_1}) \approx \sqrt{\varepsilon} e^{\omega_0 t_1}.$$

(We can choose here an arbitrary value α_0, although a small one, to be large compared to $\sqrt{\varepsilon}$, so that the condition $e^{-\omega_0 t_1} \ll e^{\omega_0 t_1}$ is fulfilled). Therefore $\omega_0 t_1 = \ln(\alpha_0/\sqrt{\varepsilon})$.

2. Oscillatory motion ($E = E_{\rm max} - \Delta E$) along the curve *2* from α_0 up to the extreme point $\alpha_{\rm m}$. Let $t = 0$ be the moment of maximal deflection, when the phase curve intersects the abscissa axis: $\dot\alpha(0) = 0$. Hence in Eq. (26) $C_2 = C_1$. Then from Eq. (24) $\alpha(0) = \alpha_{\rm m} = 2\sqrt{\varepsilon}$, and $C_1 = \sqrt{\varepsilon}$. To determine duration t_2 of this motion, we assume in Eq. (26) $\alpha(t_2) = \alpha_0$. Hence

$$\alpha_0 = \sqrt{\varepsilon}(e^{\omega_0 t_2} + e^{-\omega_0 t_2}) \approx \sqrt{\varepsilon} e^{\omega_0 t_2},$$

and we find $\omega_0 t_2 = \ln(\alpha_0/\sqrt{\varepsilon})$.

We see that $t_2 = t_1$ if $\varepsilon = \Delta E/E_{\rm max}$ is the same in both cases. Therefore the period of oscillations is twice the period of rotation for the values of energy which differ from the critical value $2mga$ on both sides by the same small amount ΔE. Indeed, we can assume with great precision that the motion from $\varphi = 0$ up to $\varphi_0 = \pi - \alpha_0$ lasts the same time t_0 given by Eq. (22), since these parts of both phase trajectories very nearly coincide with the separatrix. In the case of rotation, the remaining motion from φ_0 up to the inverted position also lasts the same time as, in the case of oscillations, does the motion from φ_0 up to the utmost deflection $\varphi_{\rm m}$, since $t_1 = t_2$.

The period of rotation $T_{\rm rot}$ is twice the duration $t_0 + t_1$ of motion from the equilibrium position $\varphi = 0$ up to the $\varphi = \pi$. Using the above value for t_1 and Eq. (22) for t_0, we find:

$$T_{\rm rot} = 2(t_0 + t_1) = \frac{2}{\omega_0}\ln\frac{4}{\sqrt{\varepsilon}} = \frac{1}{\pi}\,T_0\ln\frac{4}{\sqrt{\varepsilon}}.$$

We note that an arbitrarily chosen angle α_0 (however, $\sqrt{\varepsilon} \ll \alpha_0 \ll 1$), which delimits the two stages of motion (along the separatrix, and near the saddle point in the phase plane), falls out of the final formula for the period (when we add t_0 and t_1). The period of revolutions tends to infinity when $\varepsilon \to 0$, that is, when the energy tends to its critical value $2mga$. For $\varepsilon = 0.0001$ (for $E = 1.0001 E_{\rm max}$) the above formula gives the value $T_{\rm rot} = 1.907\,T_0$, which coincides with the cited experimental result.

The period of oscillations T is four times greater than the duration t_0+t_2 of motion from $\varphi=0$ up to the extreme point φ_m:

$$T = 4(t_0+t_2) = \frac{4}{\omega_0}\ln\frac{4}{\sqrt{\varepsilon}} = \frac{2}{\pi}T_0\ln\frac{8}{\alpha_\mathrm{m}}.$$

For $\alpha_\mathrm{m} \ll 1$ ($\varphi_\mathrm{m} \approx \pi$) this formula agrees well with the experimental results: it yields $T = 5.37\,T_0$ for $\varphi_\mathrm{m} = 179.900°$, $T = 6.83\,T_0$ for $\varphi_\mathrm{m} = 179.990°$, and $T = 8.30\,T_0$ for $\varphi_\mathrm{m} = 179.999°$. From the obtained expressions we see how both the period of oscillations T and the period of rotation T_rot tend to infinity as the total energy approaches $E_\mathrm{max} = 2mga$.

8 The Mean Energies

In the motion under consideration (swinging or rotation with $E \approx 2mga$) both potential and kinetic energies oscillate between zero and the same maximal value, which is equal to the total energy $E \approx 2mga$. However, during almost all the period the pendulum moves very slowly in the vicinity of the inverted position, and during this time its potential energy has almost the maximal value $2mga = 2J\omega_0^2$. Only for a short time, when the pendulum passes rapidly along the circle and through the bottom of the potential well, is the potential energy of the pendulum converted into kinetic energy. Hence, on the average, the potential energy predominates.

We can estimate the ratio of the averaged over a period values of the potential and kinetic energies if we take into account that most of the time the angular velocity of the pendulum is nearly zero, and for a brief time of motion the time dependence of $\varphi(t)$ is very nearly the same as it is for the limiting motion along the separatrix. Therefore we can assume that during an impulse the kinetic energy depends on time in the same way it does in the limiting motion. This assumption allows us to extend the limits of integration to $\pm\infty$. Since two sharp impulses of the angular velocity (and of the kinetic energy) occur during the period T of oscillations, we can write:

$$\langle E_\mathrm{kin}\rangle = \frac{J}{T}\int_{-\infty}^{\infty}\dot\varphi^2(t)dt = \frac{J}{T}\int_{-\pi}^{\pi}\dot\varphi(\varphi)d\varphi.$$

The integration with respect to time is replaced here with an integration over the angle. The mean kinetic energy $\langle E_\mathrm{kin}\rangle$ is proportional to the area S of the phase plane bounded by the separatrix: $2\langle E_\mathrm{kin}\rangle = JS/T$. We can substitute for $\dot\varphi(\varphi)$ its expression from the equation of the separatrix, Eq. (16):

$$\langle E_\mathrm{kin}\rangle = \frac{J}{T}2\omega_0\int_{-\pi}^{\pi}\cos\frac{\varphi}{2}d\varphi = \frac{4}{\pi}J\omega_0^2\frac{T_0}{T}.$$

Taking into account that the total energy E for this motion approximately equals $2mga = 2J\omega_0^2$, and $E_\mathrm{pot} = E - E_\mathrm{kin}$, we find:

$$\frac{\langle E_\mathrm{pot}\rangle}{\langle E_\mathrm{kin}\rangle} = \frac{2J\omega_0^2}{\langle E_\mathrm{kin}\rangle} - 1 = \frac{\pi}{2}\frac{T}{T_0} - 1.$$

For $\varphi_\mathrm{m} = 179.99°$ the period T equals $6.83\,T_0$, and so the ratio of mean values of potential and kinetic energies is 9.7 (compare with the case of small oscillations for which these mean values are equal).

9 The Influence of Friction

In the presence of weak friction inevitable in any real system, the phase portrait of the pendulum changes qualitatively: The phase curves have a different topology (compare Figures 10 and 4). A phase trajectory representing the counterclockwise rotation of the pendulum sinks lower and lower toward the separatrix with each revolution. The phase curve which passed formerly along the upper branch of the separatrix does not reach now the saddle point $(\pi,\ 0)$. Instead it begins to wind around the origin, gradually approaching it. Similarly, the lower branch crosses the abscissa axis $\dot\varphi = 0$ to the right of the saddle point $(-\pi,\ 0)$, and also spirals towards the origin.

The closed phase trajectories corresponding to oscillations of a conservative system are transformed by friction into shrinking spirals which wind around a *focus* located at the origin of the phase plane. Near the focus the size of gradually shrinking loops diminishes in a geometric progression. This focus represents a state of rest in the equilibrium position, and is an *attractor* of the phase trajectories. That is, all phase trajectories of the damped pendulum spiral in toward

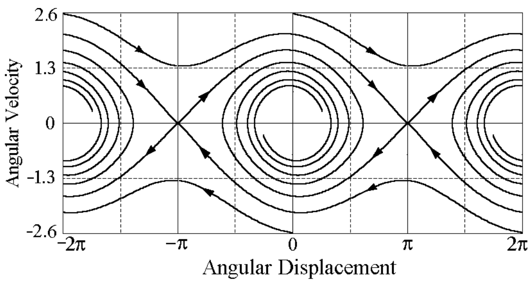

Figure 10: Phase portrait with friction.

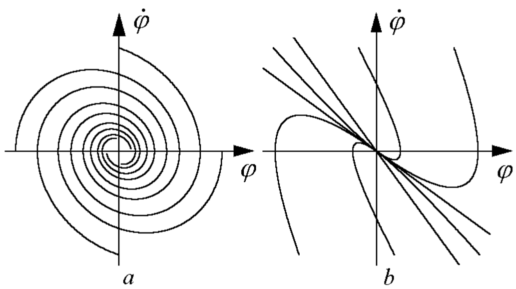

Figure 11: Phase portrait of a damped (a) and of an overdamped ($\gamma > \omega_0$, b) pendulum.

the focus, forming an infinite number of loops, as in Figure 11, a.

When friction is relatively strong ($\gamma > \omega_0$), the motion is non-oscillatory, and the attractor of the phase trajectories, instead of a focus, becomes a *node*: all phase trajectories approach this node directly, without spiraling. The phase portrait of an overdamped pendulum ($\gamma = 1.05\,\omega_0$) is shown in Figure 11,b.

When friction is weak, we can make some theoretical predictions for the motions whose phase trajectories pass close to the separatrix. For example, we can evaluate the minimal value of the initial velocity which the pendulum must be given in the lower (or some other) initial position in order to reach the inverted position, assuming that the motion occurs along the separatrix, and consequently that the dependence of the angular velocity on the angle of deflection is approximately given by the equation of the separatrix, Eq. (16).

The frictional torque is proportional to the angular velocity: $N_{\mathrm{fr}} = -2\gamma J\dot{\varphi}$. Substituting the angular velocity from Eq. (16), we find

$$N_{\mathrm{fr}} = \mp 4\gamma J\omega_0 \cos\frac{\varphi}{2} = \mp\frac{2mga}{Q}\cos\frac{\varphi}{2}.$$

Hence the work W_{fr} of the frictional force during the motion from an initial point φ_0 to the final inverted position $\varphi = \pm\pi$ is:

$$W_{\mathrm{fr}} = \int_{\varphi_0}^{\pm\pi} N_{\mathrm{fr}}\,d\varphi = -4\frac{mga}{Q}\left(1 \mp \sin\frac{\varphi_0}{2}\right). \qquad (27)$$

The necessary value of the initial angular velocity Ω can be found with the help of the conservation of energy, in which the work W_{fr} of the frictional force is taken into account:

$$\frac{1}{2}J\Omega^2 + mga(1 - \cos\varphi_0) + W_{\mathrm{fr}} = 2mga.$$

Substituting W_{fr} from Eq. (27), we obtain the following expression for Ω:

$$\Omega^2 = 2\omega_0\left[1 + \cos\varphi_0 + \frac{4}{Q}\left(1 \mp \sin\frac{\varphi_0}{2}\right)\right]. \qquad (28)$$

For $\varphi_0 \neq 0$ the sign in Eq. (28) depends on direction of the initial angular velocity. We must take the upper sign if the pendulum moves directly to the inverted position, and the lower sign if it passes first through the lower equilibrium position. In other words, at $\varphi_0 > 0$ we must take the upper sign for positive values of Ω, and the lower sign otherwise. If the pendulum is excited from the lower equilibrium position ($\varphi_0 = 0$), Eq. (28) yields the initial velocity to be

$$\Omega = \pm 2\omega_0\sqrt{1 + 2/Q} \approx \pm 2\omega_0(1 + 1/Q).$$

The exact value of Ω is slightly greater since the motion towards the inverted position occurs in the phase plane close to the separatrix but always outside it, that is, with the angular velocity of slightly greater magnitude. Consequently, the work W_{fr} of the frictional force during this motion is a little larger than the calculated value. For example, for $Q = 20$ the above estimate yields $\Omega = \pm 2.1\omega_0$, while the more precise value of Ω determined experimentally by trial and error is $\pm 2.10096\,\omega_0$.

Figure 12 shows the graphs of $\varphi(t)$ and $\dot{\varphi}(t)$ and the phase trajectory for a similar case in which the initial angular velocity is chosen exactly to let the pendulum reach the inverted position after a revolution.

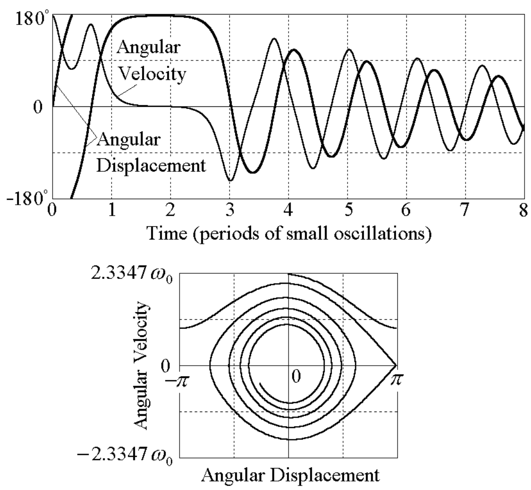

Figure 12: Revolution and subsequent oscillation of the pendulum with friction ($Q = 20$) excited from the equilibrium position with an initial angular velocity of $\Omega = 2.3347\,\omega_0$.

Summary

Free oscillations and revolutions of a rigid pendulum are investigated on the basis of a theoretical approach, aided by a computerized experimental exploration with the help of the software package "Physics of Oscillations" [1]. This package offers many interesting examples that illustrate various peculiarities of this famous physical model in vivid computer simulations, thus allowing us to appreciate the beauty of oscillatory phenomena.

The programs of the package are flexible enough and sophisticated in order to use them, say, in students' research projects for exploration of new properties of the modelled systems. Visualization of motion simultaneously with plotting the graphs of different variables and phase trajectories makes the simulation experiments very convincing and comprehensible. These simulations provide a good background for the study of more complicated nonlinear parametric systems like a pendulum whose length is periodically changed, or a pendulum with the suspension point driven periodically in the vertical direction. Such rather simple nonlinear dynamical systems demonstrate a great variety of modes in their behaviour. In particular, along with regular steady-state oscillations and synchronized rotations, these dissipative systems can exhibit examples of *dynamical chaos*, characterized by *strange attractors* in phase space. The cases of such irregular motions of systems governed by deterministic laws are distinguished by an extreme sensitivity to the initial conditions, when a very small initial difference may cause an enormous change to the long-term behaviour of the system.

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