Inertial rotation of a rigid body

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Abstract. A simple approach to the important problem of torque-free rotation of a symmetrical rigid body is suggested which is appropriate for teaching introductory mechanics and general physics to undergraduate students and is free from difficulties of traditional treatment of the problem. A small simulation program (Java-applet) is developed that visualizes the investigated motion and illustrates its principal features. The program facilitates understanding of concepts behind rigid body dynamics. Simultaneously with simulating the rigid body motion, the program presents a clear geometrical interpretation of the inertial rotation.

Introduction

Any motion of a body occurring in the absence of external forces can be called a free or inertial motion. Inertial motion is certainly the simplest possible motion. For a mass point the inertial motion is indeed a very simple one – it is a uniform rectilinear motion. However, for a rigid body only translational free motion, during which the body does not rotate, is actually simple enough. If the body rotates, its motion can be rather complicated even in the absence of external forces. The problem of torque-free rotation has many applications including any object freely falling in a gravitational field, as well as a space station or a non-spherical satellite orbiting the earth. For an axially symmetrical body, kinematics of such inertial rotation corresponds to *precession*. Being applied to a gyroscope, this free precession is called *nutation*.

The problem of a rigid body rotation has been studied over several centuries, and the equations governing the behaviour of a body have been well known since Euler. And nowadays most textbooks in physics and introductory mechanics (see, for example, [1] – [3]) still treat the problem of torque-free rotation of a rigid body on the basis of Euler equations which are referred to the non-inertial reference frame rotating together with the body. Euler equations are defined for projections of the angular velocity vector onto the coordinate axes fixed to the rotating body, and hence these equations can tell us how the momentary angular velocity behaves itself with respect to the body. However, in this problem we are interested first of all how the body moves with respect to the inertial space. For students, transition to the inertial frame of reference from a frame which rotates in a complicated manner may be rather confusing. From a pedagogical point of view, it seems expedient to find a solution to this problem directly in the inertial space.

We discuss in this paper such a solution for an axially symmetrical body in an attempt to develop some clarity regarding the problem with the help of an obvious geometrical interpretation of the investigated motion. We suggest also a simple simulation program (Java-applet) "Free rotation of an axially symmetrical body" [4] that visualizes the inertial rotation and hence can facilitate understanding about its principal features. The program is available in the Internet and is executed directly in any web browser with the Java plug-in installed.

Rotation about the principal axes of inertia

An arbitrary motion of a rigid body can be represented as a superposition of translational motion in which all points of the body, including the centre of mass, move with the same speed along parallel trajectories, and rotation around of the centre of mass. In the absence of external forces, the centre of mass moves rectilinearly and uniformly. For the analysis of rotation of a body it is expedient to use the reference frame associated with the centre of mass of the body, i.e., the inertial frame in which the centre of mass of a body is at rest and the coordinate axes have constant directions in space. In this frame of reference an arbitrary motion of a rigid body is a rotation about a fixed point (about of the centre of mass).

Kinematics of rotation of a rigid body about a fixed point is characterized by a vector $\boldsymbol{\omega}$ of momentary angular velocity. Any point of the rotating body has a (linear) velocity, which at every moment of time is exactly the same as if the body were rotating around an axis directed along the angular velocity vector. However, for the general case of free rotation, the vector of angular velocity and hence the momentary axis of rotation change continuously their direction in space. Even in the absence of external torques, that is, during the inertial rotation, behaviour of the momentary axis of rotation (sometimes this behaviour is called "wobbling") seems to be very complicated and counterintuitive. Moreover, the trajectories of different points of a freely rotating body seem even more complicated. The simulation program "Free rotation of an axially symmetrical body" illuminates various unexpected features of the inertial rotation, showing behaviour of the momentary angular velocity and the trajectory of an arbitrarily chosen point of the body. The program visualizes also the suggested geometrical interpretation of inertial rotation.

For a rotating body, vector $\boldsymbol{L}$ of the total angular momentum is proportional to the momentary angular velocity $\boldsymbol{\omega}$, but generally deviates from $\boldsymbol{\omega}$ in direction. Both vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ have a common direction only if the body rotates about one of the three mutually orthogonal axes called principal axes of inertia. Principal axes of inertia exist for every body. For symmetrical bodies manufactured of a homogeneous material, the principal axes of inertia coincide with the axes of symmetry. For example, the principal axes of inertia of a rectangular parallelepiped pass through its centre parallel to the edges. Moments of inertia, calculated with respect to the principal axes of inertia that pass through the centre of mass, are called central principal moments of inertia.

Inertial rotation of a rigid body about one of the principal axes of inertia is very simple. Indeed, during this rotation directions of vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ coincide, and since in the absence of external torques vector $\boldsymbol{L}$ of the angular momentum is conserved, vector $\boldsymbol{\omega}$ of the angular velocity is also conserved, i.e., its magnitude and direction in space remain constant. For this reason the principal axes of inertia are otherwise called the axes of free rotation. If a rigid body is set spinning about one of these axes and then released, the body simply continues spinning uniformly around the axis whose direction in space and in the body does not change. All points of the body synchronously trace circles whose centres are located on this axis.

Angular momentum and angular velocity of a symmetrical body

If the initial angular velocity deviates from a principal axis of inertia, inertial rotation of a body generally is rather complicated. Such a rotation is comparatively simple for an axially symmetrical body (a "symmetrical top"), in which two of three principal moments of inertia are equal. We emphasize that the body should not be necessarily a body of rotation – an axis of symmetry whose order is higher than two will do perfectly well to make equal all transverse moments of inertia, i.e., moments of inertia about arbitrary axes perpendicular to the axis of symmetry. For instance, a rod or bar with a square cross section, as well as any prism or pyramid whose base is a regular polygon (including triangle), manufactured of a homogeneous material, give examples of bodies which in respect of rotation are dynamically equivalent to a circular cylinder, disc or a cone, or an ellipsoid of rotation (oblate or prolate spheroid), and so on.

When such bodies are set into rotation about the axis of symmetry, vector $\boldsymbol{L}$ of the angular momentum is also directed along this axis. If vector $\boldsymbol{\omega}$ of the angular velocity deviates from the axis of symmetry through some angle, directions of vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ do not coincide anymore, but vector $\boldsymbol{L}$ necessarily lies in the same plane with $\boldsymbol{\omega}$ and the axis of symmetry. Mutual disposition of these vectors for an axially symmetrical body is shown in figure 1.

For a body whose central principal moment of inertia about a transverse axis is greater than the moment of inertia about the axis of symmetry, vector $\boldsymbol{L}$ of the angular momentum deviates from the axis of symmetry through a greater angle than vector $\boldsymbol{\omega}$ of the angular velocity does. Such mutual disposition of vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ with respect to the axis of the body is characteristic of prolate, stretched bodies (left-hand side of figure 1). For an oblate body, squeezed along the axis, vector $\boldsymbol{L}$ of the angular momentum deviates from the axis of symmetry through a smaller angle than vector $\boldsymbol{\omega}$ does (see right-hand side of figure 1).

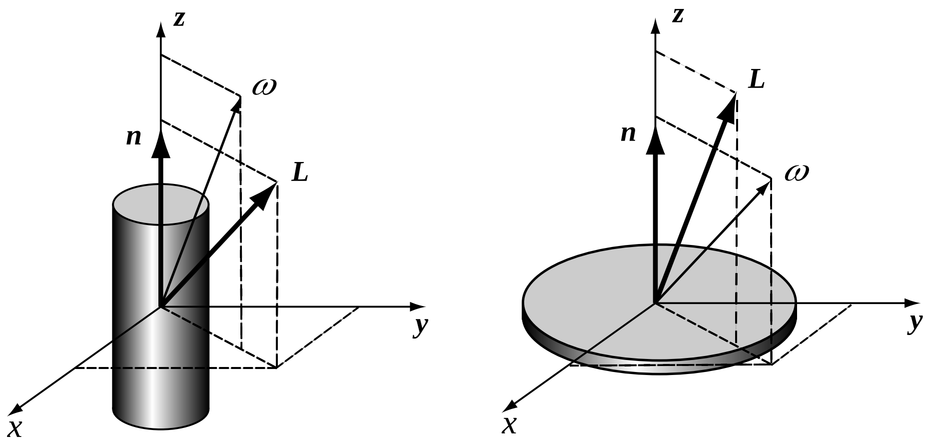

Figure 1. Mutual disposition of vector $\boldsymbol{\omega}$ of the angular velocity, vector $\boldsymbol{L}$ of the angular momentum, and the axis of symmetry (vector $\boldsymbol{n}$) for symmetrical bodies of prolate (left) and oblate (right) shapes.

Next we introduce a unit vector $\boldsymbol{n}$ to show in space the direction of the symmetry axis of our body, i.e., $\boldsymbol{n}$ is the vector starting at the origin (at the centre of mass) and directed along the axis of the body. At any moment of time all the three vectors $\boldsymbol{n}$, $\boldsymbol{L}$, and $\boldsymbol{\omega}$ lie in one plane, and their mutual disposition doesn't change during the motion of the body. The plane that contains the three vectors $\boldsymbol{n}$, $\boldsymbol{L}$, and $\boldsymbol{\omega}$, is revolving uniformly in space about vector $\boldsymbol{L}$ of the angular momentum whose direction in space remains constant in the absence of external torques. Indeed, linear velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{n}/\mathrm{d}t$ of the end point of vector $\boldsymbol{n}$, in terms of the momentary angular velocity $\boldsymbol{\omega}$ and vector $\boldsymbol{n}$ itself, is given by the formula $\mathrm{d}\boldsymbol{n}/\mathrm{d}t = \boldsymbol{\omega} \times \boldsymbol{n}$. This means that at any time moment the end point of vector $\boldsymbol{n}$ is moving perpendicularly to the plane of vectors $\boldsymbol{n}$ and $\boldsymbol{\omega}$, involving this plane together with these vectors into a uniform rotation about $\boldsymbol{L}$. During this rotation of the plane, vector $\boldsymbol{L}$ remains constant both in direction and magnitude, while vectors $\boldsymbol{n}$ and $\boldsymbol{\omega}$ move synchronously preserving their magnitudes: they describe in space circular cones with the common axis directed along $\boldsymbol{L}$. The steady motion described above is commonly called *regular precession* of vectors $\boldsymbol{n}$ and $\boldsymbol{\omega}$ about $\boldsymbol{L}$. The motion of the body itself, which consists of spinning about the own axis with simultaneous precession of this axis (revolution of the axis about $\boldsymbol{L}$) is also called a uniform or regular precession.

Regular precession of vectors $\boldsymbol{n}$ and $\boldsymbol{\omega}$ about $\boldsymbol{L}$ can be characterized by vector $\boldsymbol{\Omega}$ of the angular velocity of precession. Angular velocity of precession is directed along and proportional in magnitude to the total angular momentum $\boldsymbol{L}$, and inversely proportional to the transverse moment of inertia $I_\perp$ (moment of inertia about an axis, perpendicular to the axis of symmetry): $\boldsymbol{\Omega} = \boldsymbol{L}/I_\perp$. This will be proved below (see section *Angular velocity of precession*). With the help of $\boldsymbol{\Omega}$, for the time rate of vector $\boldsymbol{n}$ variation in space, alongside the expression $\mathrm{d}\boldsymbol{n}/\mathrm{d}t = \boldsymbol{\omega} \times \boldsymbol{n}$, we can write also $\mathrm{d}\boldsymbol{n}/\mathrm{d}t = \boldsymbol{\Omega} \times \boldsymbol{n}$. Similar expression $\mathrm{d}\boldsymbol{\omega}/\mathrm{d}t = \boldsymbol{\Omega} \times \boldsymbol{\omega}$ is valid for the time rate of vector $\boldsymbol{\omega}$ variation – both vectors $\boldsymbol{n}$ and $\boldsymbol{\omega}$ synchronously execute uniform precession about a constant vector $\boldsymbol{L}$ with the same angular velocity $\boldsymbol{\Omega}$.

When such torque-free regular precession occurs with a gyroscope and superimposes on its forced precession, it is usually called *nutation*. We note again that a torque-free precession of an object occurs if the angular velocity deviates from the axis of symmetry. The axis of a freely rotating symmetrical body preserves its direction in space (does not precess) if the angular velocity is directed along the axis of symmetry, i.e., if the body is spinning about its axis of symmetry.

A geometrical interpretation of free precession

Figure 2 shows a plain geometrical interpretation of a symmetrical body behaviour during inertial rotation. To make the presentation as clear as possible, vector $\boldsymbol{L}$ of the total angular momentum whose magnitude and direction in space are preserved in the absence of external torques, in figure 2 is oriented vertically (along z-axis).

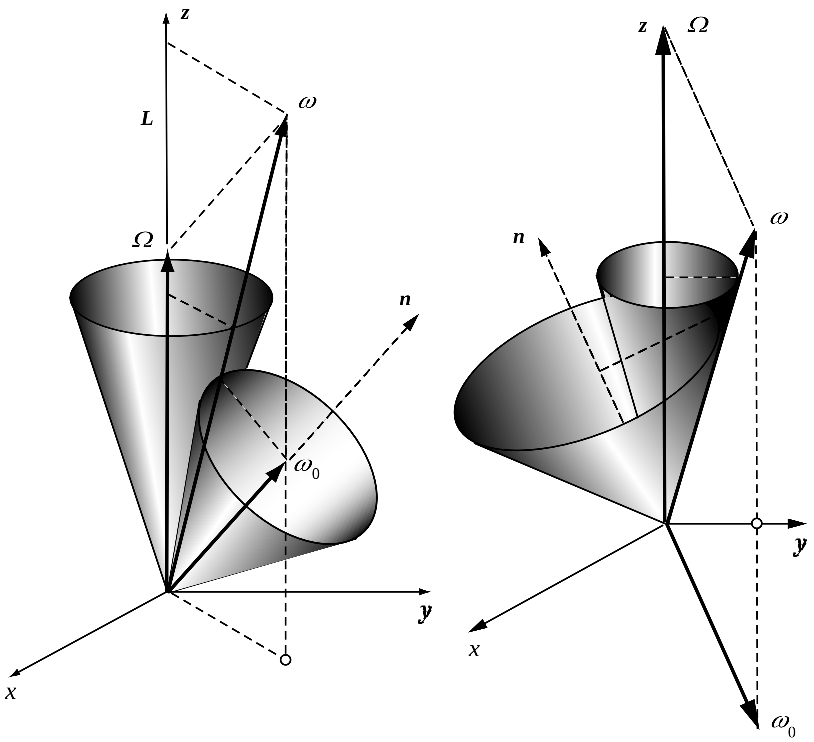

Figure 2. Geometrical interpretation of a regular precession as rolling without slipping of an imaginary moving cone attached to the body over the surface of an immovable space cone ($\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Omega}$).

At any moment of time vector $\boldsymbol{\omega}$ of the angular velocity shows how the momentary axis of rotation is directed in space. Precession of this vector about $\boldsymbol{L}$ with angular velocity $\boldsymbol{\Omega} = \boldsymbol{L}/I_\perp$ generates in space a circular cone whose vertex is located at the centre of mass of the body. The surface of this cone is formed by the set of momentary axes as they are oriented in space at different moments of time. For this reason this imaginary “space cone” (the vertical one in figure 2) is called an *immovable axoid*. The semi-angle at the vertex of this cone equals the constant angle of deviation of $\boldsymbol{\omega}$ from the fixed direction of $\boldsymbol{L}$.

Next we imagine one more circular cone, this time attached firmly to the body. The vertex of this cone is also located at the centre of mass, but its axis is directed along the axis of symmetry of the body. Let the generator of this cone be the vector of angular velocity $\boldsymbol{\omega}$, that is, the momentary axis of rotation. In other words, the lateral surface of this cone associated with the body is also formed by the set of momentary axes of rotation at different moments of time, but, contrary to the space cone, this “body cone” shows how all these axes are located inside the body (relative to the body) at different moments of time. For this reason, this imaginary cone, associated with the rotating rigid body, is called a *moving axoid*.

The moving and immovable cones touch one another by their lateral surfaces (outwardly for a prolate body, whose transverse moment of inertia is greater than longitudinal, as shown in the left-hand part of figure 2) along the momentary axis of rotation $\boldsymbol{\omega}$. All the points of the body, which are located at a given moment on the momentary axis of rotation, have zero linear velocities. This means that the moving cone (attached to the body) is *rolling without slipping* over the surface of the immovable cone, the line of contact being $\boldsymbol{\omega}$. A complicated motion of an arbitrary point of the body corresponds to addition of two rather simple motions of the body superimposed upon one another, namely, a uniform rotation about its axis of symmetry with angular velocity $\boldsymbol{\omega}_0$ and a uniform precession of this axis in space along a cone with angular velocity $\boldsymbol{\Omega}$. This geometrical interpretation of kinematics of the torque-free precession (of the inertial rotation) is clearly shown in figure 2.

We can also compare this evident geometrical interpretation of the torque-free precession (as rolling without slipping of the body cone over the surface of the space cone) with the decomposition of vector $\boldsymbol{\omega}$ of the total momentary angular velocity onto the vector sum of two components $\boldsymbol{\omega}_0$ and $\boldsymbol{\Omega}$, corresponding to the individual rotations (see figure 2):

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Omega}.$$

One of these components $\boldsymbol{\omega}_0$ corresponds to the axial rotation (spinning) of the body about its axis of symmetry. This component of the angular velocity is directed along the axis of symmetry, that is, its direction inside the body does not change during the motion. In space, this component generates a circular cone together with the axis of the body, changing with time according to the expression $d\boldsymbol{\omega}_0/dt = \boldsymbol{\Omega} \times \boldsymbol{\omega}_0$. The other component $\boldsymbol{\Omega}$, on the contrary, does not change its direction in space: this component corresponds to precession of the body axis of symmetry about angular momentum $\boldsymbol{L}$, whose direction in space (vertical on the screen) is preserved in the absence of external torques. This uniform precession occurs simultaneously with the axial rotation of the body.

Angular velocity of precession

As we already mentioned, angular velocity of precession $\boldsymbol{\Omega}$ can be expressed in terms of total angular momentum $\boldsymbol{L}$ and the transverse moment of inertia $I_\perp$ (moment of inertia about an axis perpendicular to the axis of symmetry). To find this expression, we use figure 3, in which vector $\boldsymbol{\omega}$ of the momentary angular velocity is presented as the sum $\boldsymbol{\omega}_0 + \boldsymbol{\Omega}$ of angular velocities of spin and precession, and also as the sum of mutually orthogonal longitudinal and transverse ($\omega_\perp$) components.

These two different decompositions of vector $\boldsymbol{\omega}$ correspond to two different possibilities of representing the complex torque-free rotation of a rigid body as a superposition of two simple rotations. For the first possibility ($\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Omega}$) one rotation occurs about the direction of $\boldsymbol{n}$ fixed in the body, while the other – about the direction of $\boldsymbol{L}$ fixed in space. Angular velocities $\boldsymbol{\omega}_0$ and $\boldsymbol{\Omega}$ corresponding to these rotations are not orthogonal. For the second possibility one rotation also occurs about the direction of $\boldsymbol{n}$ (but with an angular velocity different from $\boldsymbol{\omega}_0$), while the other – about a direction perpendicular to the body axis, which is also fixed in the body. In this case the two components of $\boldsymbol{\omega}$ are mutually orthogonal.

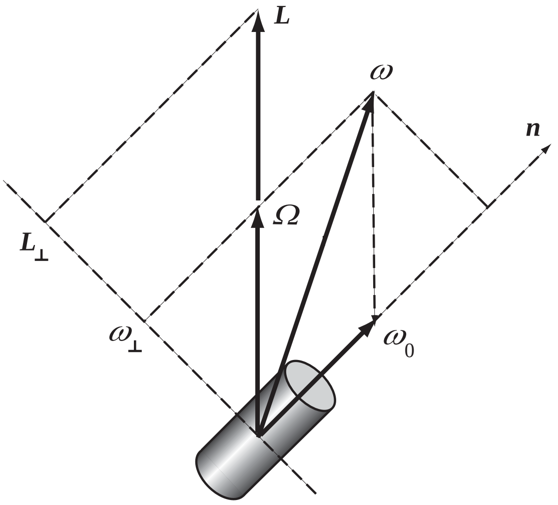

Figure 3. Relationship between vector $\boldsymbol{L}$ of the total angular momentum and vector $\boldsymbol{\Omega}$ of the angular velocity of precession.

We note that projections of both $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ onto the perpendicular axis are equal to $\omega_\perp$. Considering similar right-angle triangles with a common angle, whose hypotenuses are formed by vectors $\boldsymbol{L}$ and $\boldsymbol{\Omega}$, while legs by $\boldsymbol{L}_\perp$ and $\boldsymbol{\omega}_\perp$ respectively (see figure 3), we can write a proportion $\boldsymbol{L}/L_\perp = \boldsymbol{\Omega}/\omega_\perp$. Taking into account that $L_\perp = I_\perp\omega_\perp$, we get from this proportion the desirable expression for angular velocity $\boldsymbol{\Omega}$ of precession: $\boldsymbol{\Omega} = \boldsymbol{L}/I_\perp$. For small deviations of $\boldsymbol{\omega}$ from $\boldsymbol{n}$ (when $\omega_\perp << \omega$) this exact expression yields for the angular velocity of precession a simple approximate relationship, according to which Ω equals ω times the ratio of longitudinal and transverse moments of inertia. This means that for prolate bodies $\Omega < \omega$ (precession goes slower than the axial rotation), while for oblate bodies $\Omega > \omega$. In particular, for a thin disc the angular velocity of precession (of "wobbling") is twice as great as the angular velocity of revolution: $\Omega \approx 2\omega$.

Computer simulation of the torque-free precession

Figure 4 shows the window of the program "Free rotation of an axially symmetrical body" in which we can see the simulation of a torque-free rotation of an elongated (prolate) symmetrical top (left-hand side) together with the geometrical interpretation of this precession as rolling without slipping of the body cone over the space cone (right-hand side).

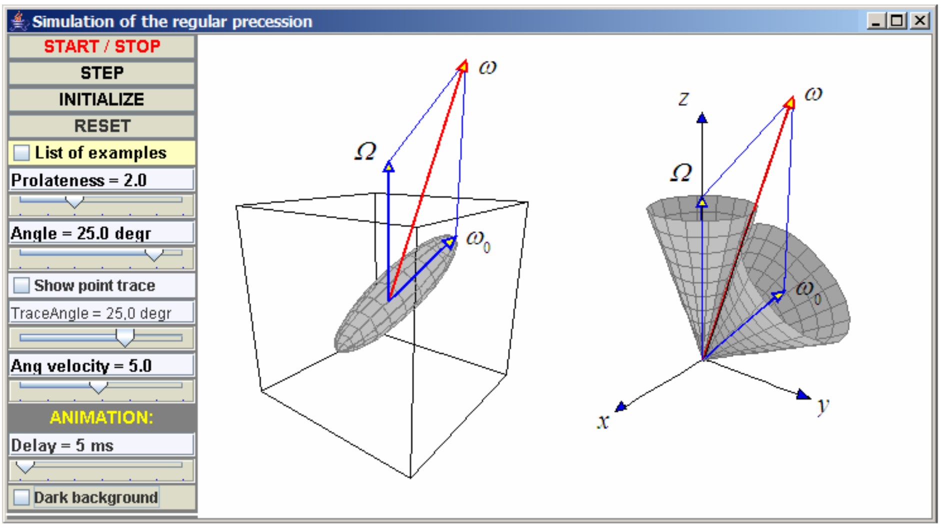

Figure 4. Simulation of the inertial rotation of an elongated symmetrical body (left) and the geometrical interpretation of this precession as rolling without slipping of the body cone over the surface of the space cone.

The simulation shows also vectors ω and ω_0, precessing with angular velocity Ω about the fixed in space direction of the angular momentum. Choosing different values for the ratio of transverse and longitudinal moments of inertia (in the program this parameter is called "Prolateness"), and different deviations of the total angular velocity ω from the body axis (parameter "Angle"), we can see clearly how vectors ω_0 and Ω add in various cases to form vector ω of momentary angular velocity.

For an oblate axially symmetrical body whose longitudinal moment of inertia is greater than the transverse one, the torque-free precession may occur even more surprising. In this situation vectors $\boldsymbol{n}$ and ω at any moment of time are deviated from vector $\boldsymbol{L}$ of the angular momentum to the opposite sides, as we can conclude from the right-hand side of figure 1. In this case the angle between vectors ω_0 and Ω is obtuse.

In other words, vector ω_0 of the spin angular velocity is directed oppositely to vector $\boldsymbol{n}$, in contrast to the case of a prolate body. This means that when the axis of the body is precessing counterclockwise and the body cone is rolling inwardly over the space cone (touching the space cone by its inner surface), the own rotation of the body occurs clockwise, that is, in the opposite sense with respect to precession. This is a retrograde kind of precession. Figure 4 gives an impression how the program simulates this behaviour.

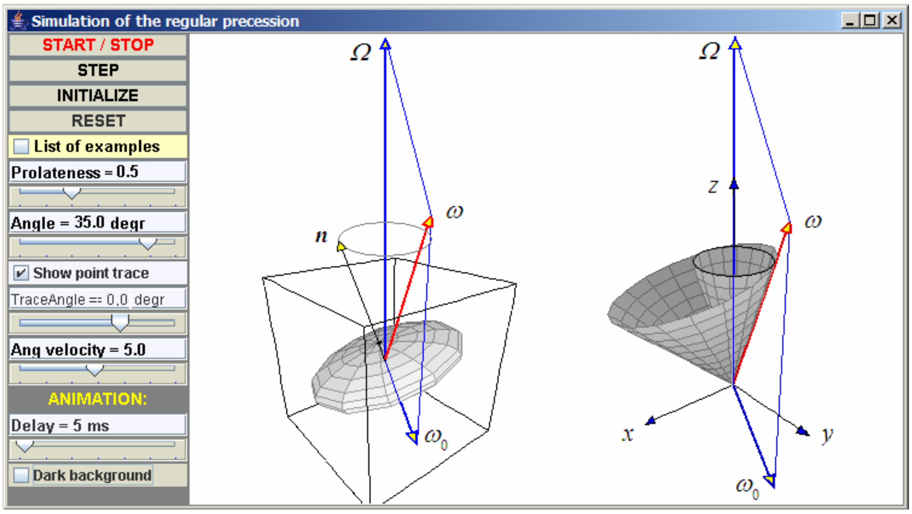

Figure 5. Computer simulation of the torque-free precession of an oblate symmetrical body (left-hand side) and the geometrical interpretation of this retrograde precession (right-hand side).

Long ago this kind of precession surprised the famous physicist Richard Feynman and stimulated him to develop a deep insight into the phenomenon. He tells the following story [5]: "I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling. I had nothing to do, so I start figuring out the motion of the rotating plate. I discovered that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation! I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there's the Dirac equation in electrodynamics. And then quantum electrodynamics. And before I knew it ... the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate."

As we already mentioned, for a thin disc the angular velocity of precession is approximately twice as great as the angular velocity of revolution: $\Omega \approx 2\omega$. Also our simulation shows clearly that at slight angles for a plate (i.e., for a thin disc) vector ω_0 of the spin angular velocity and vector $\boldsymbol{\omega}$ of the momentary angular velocity are very nearly equal in magnitude (though directed almost oppositely), while the precession (or "wobble") rate Ω is twice as fast as the spin ω_0 of the plate. It's just the opposite to Feynman's description of the wobbling plate!

Points of the body located on its axis of symmetry trace circular paths whose centres lie on the axis of the immovable cone, that is, on the axis of precession (vertical on the screen). We can consider complicated motion of points that doesn't lie on the axis of symmetry as a superposition of two relatively simple motions, namely of a rotation about the axis of symmetry with simultaneous motion of this axis in space (precession) along the surface of a vertical cone. The program, alongside the simulation of rolling without slipping of a body cone over the space cone, can show trajectories traced by such points in the course of inertial rotation. First we choose position of a point by indicating the angle between the axis of the body and direction to the point in question. For more convenient observation, the program shows the trajectory traced by the spike of an imaginary long arrow fixed to the body. The spike of this arrow lies beyond the surface bounding the body. All points of this arrow (starting at the centre of mass) trace similar trajectories, but the path of its end point shows their peculiarities in a larger scale.

In particular, we can choose a point on the lateral surface of a body cone, which at the initial moment of the simulation lies on the momentary axis of rotation. To do this, for the direction to the point we should choose the angle, equal to deviation of the angular velocity from the body axis. This point traces on a spherical surface a trajectory which looks like a cycloid, whose consecutive arcs join with one another like a sharp beak with a common tangent. For a prolate body, if we choose a point located to the body axis closer than the surface of the body cone (i.e., a point inside the body cone), the point will trace a wavy (sine-like) trajectory. Points of the body located outside the body cone trace trajectories with loops. For an oblate body, on the contrary, these points trace wavy trajectories, and vice versa.

An intermediate position between the cases of prolate and oblate bodies corresponds to the *spherical top* – a body, whose all three central principal moments of inertia are equal. The shape of a spherical top shouldn't necessarily be spherical. For example, a cube manufactured of a homogeneous material is characterized by equal principal moments of inertia about the axes parallel to its edges. This means that with respect to rotation the cube is dynamically equivalent to a spherical ball. All other regular polyhedrons (tetrahedron, octahedron, dodecahedron, icosahedron) are also spherical tops – in a torque-free rotation about the centre of mass they all behave themselves in the same way.

For a spherical top, directions of the principal axes of inertia can be chosen arbitrarily – any three mutually orthogonal axes (with the origin at the centre of mass) can be regarded as principal axes. In particular, for a cube these axes should not be necessarily parallel to its edges. This means that any axis passing through the centre is an axis of free rotation. In other words, for an arbitrary direction of vector $\boldsymbol{\omega}$ of the angular velocity, vector $\boldsymbol{L}$ of the angular momentum will have the same direction: for a spherical top an inertial torque-free rotation about any axis is simply a uniform rotation, and the direction of this axis of rotation in space is constant.

The described above geometrical interpretation of inertial rotation of an axially symmetrical body is certainly applicable to the special case in which the longitudinal and transverse moments of inertia are equal, e.g., to the case of a spherical top. During a free rotation of a spherical top vector $\boldsymbol{\omega}$ of the angular velocity and the momentary axis of rotation are directed along $\boldsymbol{L}$ and preserve their direction in space – they do not precess. This means that the space cone degenerates into a ray (a half-line) directed along vector $\boldsymbol{L}$ of the total angular momentum. Illustration of this behaviour of the spherical top in the simulation program is shown in figure 6. The axis of the body cone is the axis which we have chosen (arbitrarily) in the body as its axis of symmetry.

(For a cube this axis can be directed, say, in parallel to one of its edges, or along one of its space diagonals, or can have any other fixed direction.) Rolling of the body cone over the degenerated space cone reduces to a uniform revolution of this body cone about vector $\boldsymbol{\omega}$ that passes from the vertex of the cone (centre of mass of the body) along its lateral surface. Any point of the body (e.g., the spike of an arrow fixed to the body, see figure 6) traces a circle with the centre located on the axis of rotation.

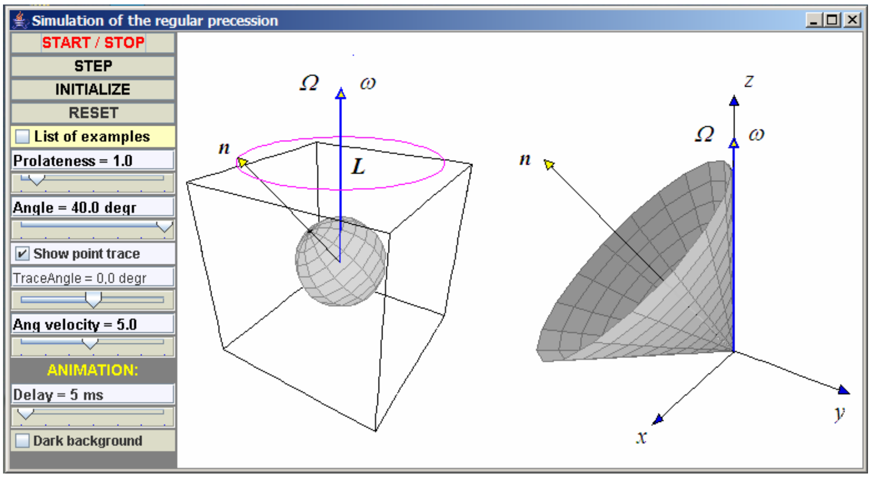

Figure 6. Torque-free rotation of a spherical top and its geometrical interpretation.

For our discussion, the special case of a spherical top is interesting in the sense that it allows us, in our geometrical interpretation of the inertial rotation, to follow the transition from outward rolling of the body cone over the space cone, occurring for prolate bodies, to the case of oblate bodies, in which the body cone touches the space cone by its inward surface (compare Figures 4 – 6).

We note that inertial rotation of a spherical top can be treated as a pure precession without any spin. Indeed, as we can clearly see from figure 6, in this case some axis of the body (arbitrarily chosen) undergoes a uniform precession, and angular velocity $\boldsymbol{\Omega}$ of this precession equals momentary angular velocity $\boldsymbol{\omega}$, which means that $\omega_0 = 0$. However, if we choose for our symmetrical top the direction of $\boldsymbol{L}$ as an axis of symmetry, the same inertial rotation can be treated as pure spinning about this axis without any precession.

Concluding remarks

In this paper we have considered the important old problem of torque-free rotation of a symmetrical rigid body. Our treatment is appropriate for teaching introductory mechanics and general physics to undergraduate students. A small simulation program (Java-applet) [4] is developed to visualize the investigated motion. Simultaneously with simulating the body motion, the program presents a clear geometrical interpretation of the inertial rotation. The suggested approach is free from difficulties inherent to traditional treatment of the problem which has many applications including objects falling in a gravitational field, non-spherical satellites orbiting the earth, etc. An understanding about the inertial (torque-free) rotation of an axially symmetrical body is also an important prerequisite for the study of a counterintuitive behaviour of a gyroscope, whose torque-induced precession is generally complicated by a nutation [6].

References

1. Kittel Ch, Knight W D, Ruderman M A 1965 – 1971 *Mechanics. Berkeley Physics Course* (New York: McGraw-Hill).
2. Goldstein H 1980 *Classical Mechanics.* 2nd ed (Addison – Wesley).
3. Goldstein H, Poole Ch P, Safko J L 2002 *Classical Mechanics.* 3rd ed (Addison – Wesley).

4. Butikov E 2006 *Free Rotation of an Axially Symmetrical Body* http://www.ifmo.ru/butikov/Applets/Precession.html
5. Feynman R P 1985 *Surely You Are Joking, Mr. Feynman!* (W W Norton). See pp. 157 – 158 for a discussion of the rotating plate motion.
6. Butikov E 2006 *Forced Precession of a Gyroscope* http://www.ifmo.ru/butikov/Applets/Gyroscope.html