INSTITUTE OF PHYSICS PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **25** (2004) L41–L43 PII: S0143-0807(04)78802-9

LETTERS AND COMMENTS

Comment on 'Eccentricity as a vector'

Eugene I Butikov

St Petersburg State University, St Petersburg, Russia

E-mail: butikov@spb.runnet.ru

Received 5 April 2004
Published 28 May 2004
Online at stacks.iop.org/EJP/25/L41
DOI: 10.1088/0143-0807/25/4/L02

Abstract

The simple derivation of the planetary orbit described recently by Bringuier is related to previous publications in this journal that deal with the orbital motion and circular hodograph in the velocity space.

Traditional derivations of a planetary orbit in the university courses of mechanics are based usually on the conservation laws of the angular momentum and the total energy (see, for example, [1]), or on a transformation of the differential equation of motion by introducing another unknown function $1/r$ instead of $r(\theta)$ [2]. For most undergraduate students the first way requires severe struggling through mathematics, while the second may seem rather artificial.

In a recent contribution to this journal Bringuier [3] suggests a very laconic and elegant way to the polar equation of the orbit, $r = p/(1 + e\cos\theta)$. In this comment, we would like to emphasize that the derivation described in [3] proves simultaneously the circularity of the velocity hodograph for any orbit in an inverse square central field (the property not mentioned in [3]), and relate the approach of [3] to the previous papers published in this journal that deal with the orbital motion.

The derivation in [3] is rather straightforward. It starts with Newton's law of motion for $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$,

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{GM}{r^2}\boldsymbol{u}_r, \tag{1}$$

where $\boldsymbol{u}_r = \boldsymbol{r}/r$ is the unit radial vector (see the left-hand part of figure 1). Then r^2 is eliminated from equation (1) with the help of the angular momentum conservation $L = mr^2\dot{\theta}$, which yields the following equation:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{GM}{C}\dot{\theta}\boldsymbol{u}_r. \tag{2}$$

Here a constant $C = L/m = r^2\dot{\theta}$ is introduced whose meaning is the angular momentum magnitude per unit mass, or the sectorial velocity doubled ($C = 2\,\mathrm{d}S/\mathrm{d}t$).

The crucial point of the derivation is the substitution of $\mathrm{d}\boldsymbol{u}_\theta/\mathrm{d}t$ in equation (2) instead of $-\dot{\theta}\boldsymbol{u}_r$ with subsequent straightforward integration with respect to time t, which gives the

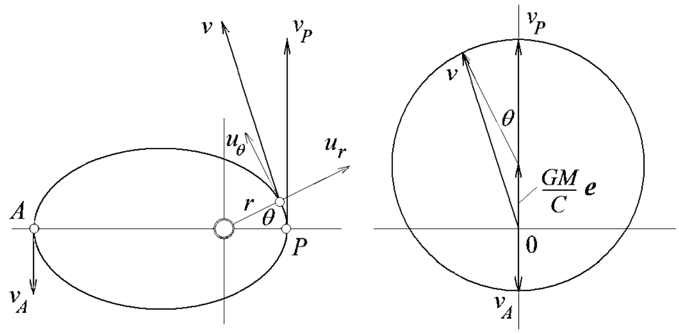

Figure 1. Keplerian orbit and the velocity vectors in space (left), and the hodograph of the velocity vector in velocity space (right).

expression for vector $\boldsymbol{v}$ as the sum of two vectors:

$$\boldsymbol{v} = \frac{GM}{C}(\boldsymbol{u}_\theta + \boldsymbol{e}). \tag{3}$$

The second term on the right-hand side of equation (3) (the constant of integration) is a time-independent vector of magnitude $(GM/C)e$, while the first one is a vector of constant magnitude (GM/C) pointing currently in the direction of the unit vector $\boldsymbol{u}_\theta$, which is perpendicular to the momentary radius vector $\boldsymbol{r}$.

Equation (3), which is exactly equation (5) of [3], proves actually that the velocity hodograph for an arbitrary Keplerian motion is a circle (see the right-hand side of figure 1). Indeed, the unit vector $\boldsymbol{u}_\theta$ changes its direction as the body moves along its orbit, and hence the vector $(GM/C)\boldsymbol{u}_\theta$ of fixed magnitude GM/C rotates (non-uniformly) in velocity space about the point to which the constant vector $(GM/C)\boldsymbol{e}$ points. Since $\boldsymbol{v}$ is the sum of these two vectors, its end generates the same circle (or an arc of the circle for hyperbolic orbits). This statement is equally valid for all closed (elliptical) and open (parabolic and hyperbolic) orbits traced under the inverse square central force.

A similar, although less straightforward derivation of the circular shape of the velocity hodograph (based primarily on geometrical considerations) can be found in [4]. The velocity vector of a body in an arbitrary Keplerian motion is represented in [4] also as the sum of two vectors ($\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ in the notation of [4]), one of which ($\boldsymbol{w} = (GM/C)\boldsymbol{e}$) points always from the origin to the same point of velocity space (the centre of the hodograph), while the other vector of a constant magnitude ($\boldsymbol{u} = (GM/C)\boldsymbol{u}_\theta$) generates the circle. This interesting property of an arbitrary Keplerian motion is very clearly illustrated by the simulation program [6].

Equation (3) of this paper and equation (5) in [3] coincide with equation (3) of [5] also derived in [5] from Newton's second law and the conservation of angular momentum. The mentioned equations in the cited papers differ only in notation. However, when the constant component $\boldsymbol{w}$ of the velocity vector (that points to the centre of the hodograph) is denoted by $(GM/C)\boldsymbol{e}$ (as in [3]), the magnitude e of vector $\boldsymbol{e}$ has the clear physical meaning of the eccentricity of the orbit. This makes it reasonable to call $\boldsymbol{e}$ the *eccentricity vector*.

The constant magnitude GM/C of the other vector $\boldsymbol{u} = (GM/C)\boldsymbol{u}_\theta$ (radius of the circular hodograph) can be conveniently expressed in terms of velocity v_P at the perihelion (perigee) and the circular velocity $v_\mathrm{c} = \sqrt{GM/r_P}$ for this point P of the orbit (see [6]): $u = GM/C = v_\mathrm{c}^2/v_P$. The displacement w of the hodograph centre from the origin of the velocity space can be expressed as $v_P - v_c^2 / v_P$ or, equivalently, as $w = ue$, where the eccentricity $e = v_P^2 / v_c^2 - 1$.

The last step, which allows us to obtain the orbit from equation (3), is rather obvious: it consists in taking a projection of both sides of equation (3) on the direction of the unit vector $\boldsymbol{u}_\theta$. From the left-hand part of figure 1 we see that this projection of $\boldsymbol{v}$ equals $r\dot{\theta}$ or C/r (if we take into account that $r^2\dot{\theta} = C$). The right-hand part of the figure shows that at the same time this projection equals $(GM/C)(1 + e\cos\theta)$. Equating these values, we obtain the desired equation of the orbit:

$$r = \frac{p}{1 + e\cos\theta}, \qquad \text{where} \quad p = \frac{C^2}{GM} = \frac{L^2}{GMm^2} = r_P(1 + e). \tag{4}$$

The geometrical way from equation (3) to (4) described above, being equivalent to the derivation in [3], may seem more natural to undergraduate students. We note that the same idea is also used in [5].

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