

# Precession and nutation of a gyroscope

**Eugene Butikov**

St Petersburg State University, St Petersburg, Russia

E-mail: [butikov@spb.runnet.ru](mailto:butikov@spb.runnet.ru)

Received 24 March 2006, in final form 21 April 2006

Published 17 July 2006

Online at [stacks.iop.org/EJP/27/1071](http://stacks.iop.org/EJP/27/1071)

## Abstract

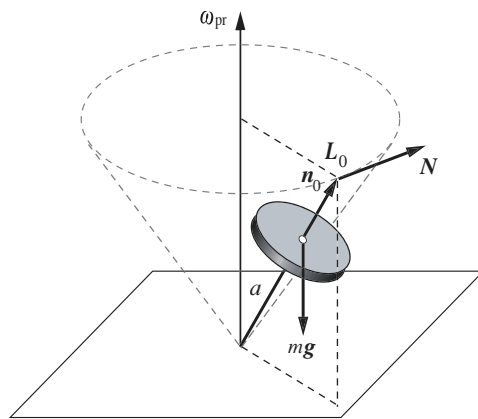
A simple treatment of the important old problem of the torque-induced rotation of a spinning symmetrical top is suggested. Our discussion is appropriate for teaching introductory mechanics and general physics to undergraduate students and is free from the difficulties of a traditional approach to the problem. The origin of nutation that accompanies forced precession is explained in detail, with an accent on its relationship to the torque-free precession of a symmetrical body. A small simulation program is developed that visualizes the investigated motion and illustrates its principal features. The program facilitates understanding about the counterintuitive behaviour of a gyroscope on a qualitative level.

(Some figures in this article are in colour only in the electronic version)

## Introduction

A gyroscope is a body of rotation (for example, a massive disc) which is set spinning at a large angular velocity around its axis of symmetry. First acquaintance with a gyroscope occurs usually in early childhood, when the child watches the unusual behaviour of a widely known toy—a spinning top. A toy gyroscope seems to be able to defy gravity. As long as the top is spinning fast enough, it remains staying steadily ('sleeping') on the lower sharp end of the axis avoiding tipping over or falling down to the ground and preserving the vertical position of the axis in spite of the high position of its centre of mass—the centre of gravity of a spinning top can be located far above its supporting pivot. If the axis of a spinning gyroscope is inclined to the vertical, the axis generates in space a vertical circular cone, so that the angle between the gyroscope axis and the vertical remains constant during rotation. This kind of motion of a gyroscope under an external torque is called forced or *torque-induced regular precession* (see figure 1). Precession is often complicated by *nutation* which reveals itself as a fast shivering of the precessing axis.

The most important practical applications of gyroscopes are met in devices for measuring the orientation or maintaining the stability of airplanes, spacecraft and submarines—vehicles in general. Various gyroscopes are used as sensors in inertial guidance systems.



**Figure 1.** Steady (regular) precession of the gyroscope under the force of gravity.

This paper is an attempt to develop some clarity regarding the old problem of precession and especially nutation of gyroscopes on the level appropriate for undergraduate students. Most textbooks in introductory mechanics (see, for example, [1, 2]) explain the mysterious behaviour of a spinning gyroscope using Lagrange equations and severe mathematics, which is not very helpful for developing physical intuition. Those textbooks that try to avoid such difficulties omit as a rule any discussion or explanation of nutation (see [3, 4]). This may form an impression that a gyroscope is some wonder that does not obey the fundamental laws of physics. Other textbooks [5, 6] treat the problem of torque-induced precession of a top on the basis of Euler equations which are referred to the non-inertial reference frame rotating together with the body. However, in this problem we are interested first of all how the body moves with respect to the inertial space. For students, transition to the inertial frame of reference from a body frame which rotates in a complicated manner may be rather confusing. From a pedagogical point of view, it seems expedient to find a solution to this problem directly in the inertial space.

In addition, a small simulation program (Java applet) ‘Forced Precession of a Gyroscope’ [7] (with a commenting text) is developed to visualize and help students grasp and better understand on a qualitative level a somewhat counterintuitive behaviour of a gyroscope, including precession under the force of gravity and nutation. The program is available on the Internet and is executed directly in any web browser with the Java plug-in installed.

### Regular precession of the gyroscope

The strange at first sight behaviour of the gyroscope can be explained on the basis of the main law of rigid body dynamics according to which the time rate  $d\mathbf{L}/dt$  of change in the angular momentum  $\mathbf{L}$  equals the total torque  $\mathbf{N} = \sum_i (\mathbf{r}_i \times \mathbf{F}_i)$  of the external forces exerted on the body:

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}. \quad (1)$$

When the gyroscope is set to rotation around its axis of symmetry, vector  $\mathbf{L}$  of its angular momentum is also directed along this axis. We denote the orientation of the top axis in space (from the pivot point towards the centre of mass) by vector  $\mathbf{n}_0$  of the unit length (see figure 1).

Both vectors  $\mathbf{L}$  and  $\mathbf{n}_0$  can point in the same or in opposite directions depending on the sense of rotation. Let  $I_0$  be the moment of inertia of the top about its axis of symmetry. In particular, for a disc or circular cylinder manufactured of a homogeneous material  $I_0 = \frac{1}{2}mR^2$ , where  $m$  is the mass of the body,  $R$  is its radius. When the top is rotating about its axis with an angular velocity  $\omega_0$  (positive for counter-clock rotation), the angular momentum can be represented as  $\mathbf{L} = I_0\omega_0\mathbf{n}_0$ .

Being inclined to the vertical, the gyroscope undergoes precession, that is, besides rotation around its own axis, it turns also around the vertical axis. At fast own rotation this precession occurs so slowly that with a good accuracy it is possible to neglect the component of the angular momentum which is caused by precession around the vertical. In other words, even in this case we can assume that vector  $\mathbf{L}$  of the total angular momentum is directed approximately along the axis of the gyroscope, i.e., along the slowly moving vector  $\mathbf{n}_0$ :  $\mathbf{L} \cong \mathbf{L}_0 = I_0\omega_0\mathbf{n}_0$ . The term ‘gyroscope’ is usually applied to an axially symmetric body only if this condition of fast axial rotation is fulfilled. An approximate theory of a gyroscope is based on the assumption that vector  $\mathbf{L}$  of the total angular momentum is always directed along the axis of symmetry. Therefore the behaviour of vector  $\mathbf{L}$  deduced from equation (1) tells us about the behaviour of the gyroscope axis.

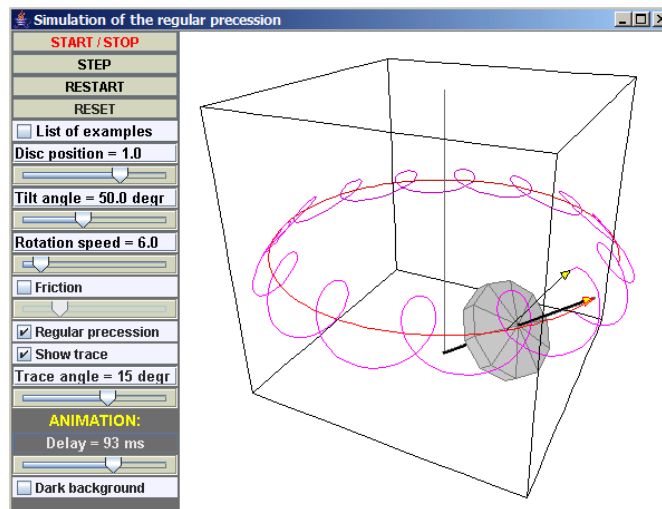
If a rigid body moves about a fixed point, it is convenient to use this point as the reference point (the origin) in the definition of angular momentum  $\mathbf{L}$  and torque  $\mathbf{N}$  in equation (1). For a toy top this point is a pivot at which the sharp lower end of the top axis is supported. When we choose this point, the unknown reaction force (exerted on the gyroscope just at this point) has zero torque on the right-hand side of (1), and therefore we can investigate the motion of the top without knowing this force. Hence for a gyroscope whose axis is supported at a point different from the centre of mass, the external torque  $\mathbf{N}$  in equation (1) about the pivot point is provided solely by the force of gravity  $m\mathbf{g}$ , which in the uniform gravitational field can be regarded as applied at the centre of mass of the top.

The position vector of the centre of mass relative to the pivot is  $\mathbf{r}_c = \mathbf{n}_0a$ , where  $a$  equals the distance between the pivot and the centre of mass. If the axis is not perfectly vertical, the torque of the force of gravity  $\mathbf{N} = \mathbf{r}_c \times \mathbf{F} = \mathbf{n}_0a \times m\mathbf{g}$  is directed at a right angle to vector  $\mathbf{L}$  of the angular momentum and thus causes it to change direction but not magnitude. At any moment of time vector  $\mathbf{N}$ , being directed as the cross product of  $\mathbf{n}_0$  and  $\mathbf{g}$ , points perpendicularly to the gyroscope axis  $\mathbf{n}_0$  and perpendicularly to  $\mathbf{g}$  (horizontally). This explains the unexpected at first sight behaviour of the gyroscope axis: according to (1), the upper end of vector  $\mathbf{L}$  moves in the direction of vector  $\mathbf{N}$  rather than in the direction of force  $m\mathbf{g}$  itself. During an infinitesimal time interval  $dt$  vector  $\mathbf{L}$  gets an elementary increment  $d\mathbf{L} = \mathbf{N}dt$  directed along  $\mathbf{N}$  and thus lying in the horizontal plane at a right angle to the gyroscope axis. This means that vector  $\mathbf{L}$  and also the axis of the top undergo a regular precession, that is, they both are uniformly revolving about the vertical line that passes through the pivot.

The angular velocity  $\omega_{\text{pr}}$  of this precession can be found by substituting  $\mathbf{L} \cong \mathbf{L}_0 = I_0\omega_0\mathbf{n}_0$  into the left-hand side of (1) and  $\mathbf{N} = \mathbf{n}_0a \times m\mathbf{g}$  into its right-hand side. If the friction is insignificant, we can consider the angular velocity  $\omega_0$  of the axial rotation to be constant. Therefore on the left-hand side of equation (1) we can differentiate only  $\mathbf{n}_0$  in  $\mathbf{L} \cong I_0\omega_0\mathbf{n}_0$ . Then equation (1) yields

$$I_0\omega_0 \frac{d\mathbf{n}_0}{dt} = \mathbf{n}_0a \times m\mathbf{g}, \quad \text{or} \quad \frac{d\mathbf{n}_0}{dt} = \omega_{\text{pr}} \times \mathbf{n}_0, \quad \omega_{\text{pr}} = -\frac{am}{I_0\omega_0}\mathbf{g}. \quad (2)$$

As follows from equation (2), at  $\omega_0 > 0$  (i.e., when the gyroscope rotates counter-clockwise about its axis) the vector  $\omega_{\text{pr}}$  of the angular velocity of precession is directed upward, oppositely to vector  $\mathbf{g}$ , which means that precession induced by gravity occurs also counter-clockwise—in the same sense as the axial rotation. And at clockwise axial rotation



**Figure 2.** Simulation of the regular precession of the gyroscope. The image shows the circular trajectory of the upper end of the gyroscope axis and the loopy trajectory of the end point of an arrow attached firmly to the gyroscope.

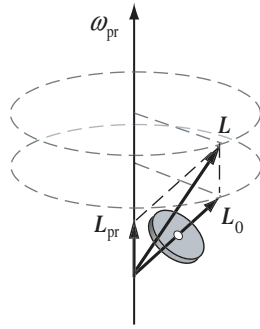
of the gyroscope (at  $\omega_0 < 0$ ) its precession forced by gravity occurs also clockwise. The magnitude  $\omega_{pr}$  of the angular velocity of precession is inversely proportional to the angular velocity  $\omega_0$  of the axial rotation and directly proportional to the distance  $a$  between the pivot and the centre of mass. We note that  $\omega_{pr}$  is independent of the tilt of the axis to the vertical.

The behaviour of the gyroscope axis described by (2) is called *regular* or *uniform precession*. This is a forced, *torque-induced precession* because the gyroscope undergoes it under the force of gravity. All points of the gyroscope that lie on its axis of symmetry move uniformly along circular paths whose centres are located on the vertical line passing through the supporting pivot. Figure 2 shows the applet window of the simulation computer program ‘Forced Precession of a Gyroscope’ [7] with an illustration of regular precession.

The program shows the circular path along which the upper end of the gyroscope axis moves (the red circle in figure 2), and also a loopy trajectory of some fixed point of the top, which is displaced from the axis of symmetry. The position of this point is indicated by an arrow which originates at the centre of mass and ends beyond the bounding surface of the top. This is done deliberately in order to show the peculiarities of these trajectories on a larger scale. We can imagine this arrow as being attached firmly to the body of the gyroscope (plunged into it up to the centre of mass). For convenience, the distance from the pivot to the end point of the arrow is chosen to be equal to the length of the top’s axis. Therefore both trajectories of the upper end of the gyroscope axis and of the arrow spike lie on the same sphere whose centre is located at the pivot.

We have considered above the torque-induced regular precession of the gyroscope on the basis of an *approximate* theory in which the total angular momentum  $\mathbf{L}$  of the gyroscope is assumed to be always directed along its axis of symmetry. However, the most remarkable feature of regular precession is that this kind of motion is actually one of the possible solutions to the *exact* dynamical equation of motion (equation (1)). Next we will show this.

In the exact theory of the gyroscope we must take into account that the total angular momentum  $\mathbf{L}$  is produced not only by rotation about the axis of symmetry. Besides this



**Figure 3.** Vector  $\mathbf{L}$  of the total angular momentum at regular precession of the gyroscope.

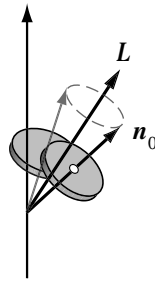
own angular momentum  $\mathbf{L}_0 = I_0\omega_0\mathbf{n}_0$ , which constitutes the major part of  $\mathbf{L}$ , some additional investment in  $\mathbf{L}$  is given by (slow) rotation about other axes. In the case of regular precession this additional angular momentum is produced by the revolution of the top about the vertical line with (small) angular velocity  $\omega_{\text{pr}}$ . Let us denote by  $\mathbf{L}_{\text{pr}}$  the corresponding vector, which is proportional to  $\omega_{\text{pr}}$  and coincides with  $\omega_{\text{pr}}$  in direction, as shown in figure 3.

When we take  $\mathbf{L}_{\text{pr}}$  into account, we realize that vector  $\mathbf{L}$  of the total angular momentum is not directed exactly along the axis of the top (see figure 3). Nevertheless, the horizontal components of vectors  $\mathbf{L}$  and  $\mathbf{L}_0$  are equal when the gyroscope undergoes a regular precession. The end points of vectors  $\mathbf{L}$  and  $\mathbf{L}_0$  move synchronously along equal circles which are shown in figure 3 by dashed lines. This means that these end points of  $\mathbf{L}$  and  $\mathbf{L}_0$  have equal velocities. In other words, the time derivatives of  $\mathbf{L}$  and  $\mathbf{L}_0$  are equal:  $d\mathbf{L}/dt = d\mathbf{L}_0/dt$ . Both vectors  $\mathbf{L}$  and  $\mathbf{L}_0$  undergo precession about the vertical with the same angular velocity  $\omega_{\text{pr}}$ , and their variation with time is described by similar equations:

$$\frac{d\mathbf{L}_0}{dt} = \omega_{\text{pr}} \times \mathbf{L}_0, \quad \frac{d\mathbf{L}}{dt} = \omega_{\text{pr}} \times \mathbf{L}. \quad (3)$$

The second of these equations testifies that regular precession with an angular velocity  $\omega_{\text{pr}}$  is actually a solution to the exact dynamical equation (1). However, this is not the only possible solution to the equation of motion (1) under the torque of the force of gravity. The regular precession of a gyroscope occurs only if the initial conditions are quite definite: in order to observe this regular behaviour, we should make the gyroscope spin around its axis and set to this axis a rotation about the vertical line with a certain angular velocity, namely the velocity which is characteristic of the subsequent precession. Only this initial angular velocity of the axis will provide the regular precession. In this sense we can say that the force of gravity that tends to tip over the gyroscope essentially only maintains, but not causes the precession—actually the regular precession is caused by these specific initial conditions.

We can compare the role which the force of gravity plays in the regular precession of a gyroscope with the role of string tension when we make a small body twirling on the end of the string. The force of string tension pulls the attached body towards the centre of the circle, but the body moves all the time under this force perpendicularly to the force. The force of string tension only maintains, but not causes the motion of the body. To produce this motion, we must impart to the body an initial velocity in the perpendicular direction. With this analogy in mind, we may consider the gyroscope behaviour at a regular precession not as strange and surprising as it may seem at first sight.



**Figure 4.** Torque-free precession (nutation) of the gyroscope about the total angular momentum  $L$  in the absence of gravity.

### Nutation of the gyroscope

In the general case, i.e., for arbitrary initial conditions, the motion of a gyroscope is a superposition of torque-induced regular precession and torque-free *nutation*. The nutation (etymologically a ‘nodding’) of a fast-spinning gyroscope reveals itself as (small) vibration and shivering of the precessing axis. Nutation is caused by a possible small deviation of the vector of own angular momentum from the axis of symmetry. This deviation is absent only for carefully chosen specific initial conditions.

In order to understand what will be the motion of the gyroscope at arbitrary initial conditions which do not provide regular precession, we consider the following auxiliary problem. Let us imagine how the gyroscope executing regular precession under gravity torque will move if at some moment of time the force of gravity would suddenly vanish. This situation can be realized in practice by letting the whole apparatus (the gyroscope together with its supporting device) fall freely in the gravitational field, as in Einstein’s thought experiments with a freely falling elevator cabin. When the cabin starts to fall, the gyroscope instantly becomes weightless.

For a possible explanation of the further motion of the gyroscope, we should turn to the problem of the *torque-free* inertial rotation of a symmetrical top. This problem is discussed, in particular, in the preceding paper [8] and illustrated by a simulation computer program ‘Free rotation of an axially symmetrical body’ [9]. The torque-free motion of the top can be represented as a superposition of two rotations: spinning about the axis of symmetry with the angular velocity  $\omega_0$  and a simultaneous uniform rotation of this axis about the direction of the angular momentum  $L$ . The body axis moves about vector  $L$  in a circular cone. The angular velocity of the torque-free precession is directed along and proportional in magnitude to the total angular momentum  $L$ , and inversely proportional to the transverse moment of inertia  $I_{\perp}$  (moment of inertia about an axis, perpendicular to the axis of symmetry):  $\Omega = L/I_{\perp}$ . Being applied to a gyroscope, this free precession is usually called nutation.

Returning to the precessing gyroscope which suddenly occurs in weightlessness, we note that at this moment its vector  $L$  of the total angular momentum does not point along the axis of symmetry. This is clearly seen in figure 3. At the time moment when the force of gravity is suddenly ‘switched off,’ the torque-induced precession of vector  $L$  immediately stops, but the gyroscope axis continues its precessing motion. The velocity of this motion should be regarded as the initial velocity for the subsequent torque-free motion of the gyroscope. A picture of this motion is shown in figure 4. This is a nutation in which the gyroscope axis (vector  $n_0$ ) describes a circular cone about vector  $L$  of the conserved total angular momentum with an angular velocity  $\Omega = L/I_{\perp}$ . Thus, if we forget for a while about the force of gravity,

the motion of the gyroscope would be a nutation, which is simply another name for an inertial (torque-free) precession.

The existence of nutation explains one more peculiarity in gyroscope behaviour which may seem to be paradoxical. According to equation (1), the angular momentum  $\mathbf{L}$  of a gyroscope can change with time only under external forces whose torque  $\mathbf{N}$  has a non-zero value. If the external forces suddenly vanish as in the example discussed above, the time variation of the angular momentum  $\mathbf{L}$  stops just at the same moment. One may think that this implies the immediate termination of the axis' motion—the gyroscope axis should stop at once. The centre of mass which during the precession was moving in a circle should also stop immediately together with the axis. Can this behaviour of the gyroscope be in agreement with the law of inertia?

Such a contradiction would actually exist if the described absence of inertia referred to the motion of the gyroscope axis rather than of vector  $\mathbf{L}$ . The angular momentum  $\mathbf{L}$  actually exhibits no inertia and, as follows from equation (1), stops to change immediately if the external torque vanishes. The fake conclusion regarding the absence of inertia in the behaviour of the axis is imposed by the approximate theory of a gyroscope in which vector  $\mathbf{L}$  of the angular momentum is supposed to be directed always along the gyroscope axis. In reality vector  $\mathbf{L}$  of the total angular momentum deviates during a torque-induced precession from the axis of the gyroscope by some angle, and if the external force vanishes, nutation starts immediately: the axis describes a cone about the final direction of  $\mathbf{L}$  which is preserved further on. The velocity of the centre of mass at the initial moment of nutation is the same as during the preceding forced precession. Therefore the contradiction with the law of inertia disappears when nutation is taken into account.

### Forced precession accompanied by a nutation

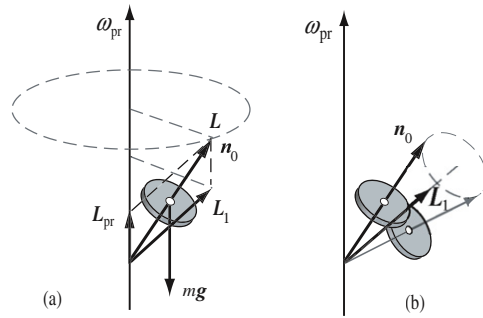
Now we are ready to investigate the problem of spinning gyroscope behaviour under gravitational torque at initial conditions which do not provide regular precession from the very beginning. The previous discussion brings us to a conclusion that in the general case gyroscope motion is a superposition of torque-induced precession and free nutation.

Next we treat for definite the most typical situation: first we set the tilted gyroscope to a fast spinning about its axis, supporting simultaneously the upper end of its axis against gravity. To do this, we exert on the axel an additional upward force. The torque of this force about the pivot compensates for the torque of the force of gravity. If we release the upper end of the axel before setting the gyroscope to spin, the force of gravity would simply tip it over, turning down in the vertical plane around the pivot with acceleration. But what will be the motion of the tilted gyroscope if we release it only after setting it into a fast spinning?

Obviously, for these initial conditions vector  $\mathbf{L}$  of the total angular momentum is directed initially along the axis. As soon as we release the axis, the torque of gravity appears, and under this torque vector  $\mathbf{L}$ , according to (1), starts to change. Its variation occurs exactly in the same way as during the regular precession. We emphasize that this precession of vector  $\mathbf{L}$  starts with a finite angular velocity  $\omega_{pr}$  immediately after we release the axis, but the axis itself and the centre of mass of the gyroscope are still immovable, that is, they have zero velocity at the initial time moment. Using these initial conditions, next we try to represent the subsequent motion of the gyroscope as a superposition of torque-induced precession and nutation.

The involvement of the gyroscope in torque-induced precession implies that vector  $\mathbf{L}$  of the total angular momentum has a vertical component  $\mathbf{L}_{pr}$  due to the revolution of the top about the vertical line with an angular velocity  $\omega_{pr}$ . Let us single out  $\mathbf{L}_{pr}$  from the total angular momentum  $\mathbf{L}$ , i.e., represent  $\mathbf{L}$  as the sum  $\mathbf{L}_{pr} + \mathbf{L}_1$  (see figure 5(a)). The remaining part  $\mathbf{L}_1$





**Figure 5.** Superimposition of nutation about the direction of  $L_1$  onto the forced precession.

can be considered as the contribution of spin (own rotation of the top) into the total angular momentum  $L$ .

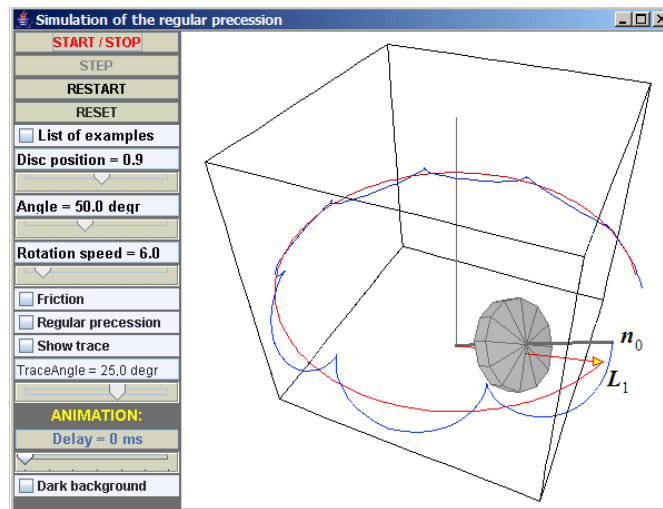
We conclude from figure 5(a) that the angular momentum  $L_1$  associated with this spinning is not directed along the gyroscope axis: vector  $L_1$  deviates from the vertical through a greater angle than the axis. Let us forget for a while about the torque-induced precession of the top and concentrate on its spinning. The deviation of vector  $L_1$  from the axis implies that simultaneously with the slow forced precession the gyroscope undergoes fast nutation, i.e., the axis describes a circular cone (with a small semi-angle) with an angular velocity  $\Omega = L_1/I_\perp$  about the slowly precessing vector  $L_1$ . Superposition of these two motions can provide the zero velocity of the axis and of the centre of mass for the initial moment, as required by the initial conditions. This requirement of initial velocity zero for the axis fulfils only if vector  $L_1$  is deviated from the gyroscope axis (from vector  $n_0$ ) through a definite angle which depends on the tilt of the axis and on the ratio of the angular velocities of nutation and the forced precession. This angle between  $L_1$  and  $n_0$  determines the semi-angle of the nutation cone.

For a fast-spinning gyroscope  $L_1 \approx I_0\omega_0$ , whence  $\Omega = L_1/I_\perp \approx (I_0/I_\perp)\omega_0$ , that is, the angular speed  $\Omega$  of nutation is of the same order of magnitude as the angular speed  $\omega_0$  of spin (provided the longitudinal and transverse moments of inertia do not differ considerably).

Thus, the axis of the top (vector  $n_0$  in figure 5), being released with zero velocity, undergoes simultaneously two rather simple motions: along the vertical cone due to the torque-induced steady (uniform) precession (figure 5(a)) and along a small cone of nutation about vector  $L_1$  of the angular momentum related to own rotation (figure 5(b)). The upper end point of the axis moves due to precession along a horizontal circle shown by a dashed line in figure 5(a); due to nutation, along the small circle whose centre lies on  $L_1$ , as shown in figure 5(b). Both these motions start at the moment of time at which the axis of the spinning top is released—before this moment the top was simply spinning about its axis. Addition of these two motions generates for the end of the axis a trajectory that reminds us of a cycloid—the curve traced by a point on the rim of a wheel rolling without slipping. The simulation program [7] allows us to see details of this three-dimensional curve (see figure 6) traced by the axis end on the surface of the sphere centred at the pivot. The red horizontal circle is the trajectory traced by the precessing vector  $L_1$ .

We note that firstly, after we release the axis, it actually starts to fall down under the influence of gravity, in accordance with our intuition, and the tilt increases. But as the top falls and gains speed, the trajectory of its end deviates more and more from the vertical. Soon the velocity of the axis becomes horizontal, just as this velocity is directed at regular precession, but its magnitude is greater than is needed for regular precession. This makes the axis move up. When the axis reaches the initial height, its velocity turns to zero. Then all the motion





**Figure 6.** Simulation of the precession induced by the force of gravity. This precession occurring after the axis of a spinning top is released with initial velocity zero is accompanied by nutation.

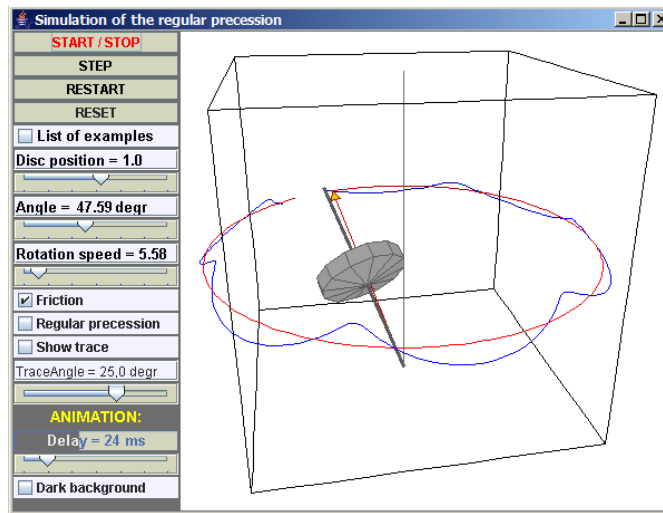
repeats. But now we realize that this strange at first sight behaviour of the axis is explained by addition of two motions—fast nutation about vector  $L_1$  due to the deviation of  $L_1$  from the axis of symmetry, and simultaneous torque-induced slow precession of this vector about the vertical.

For different initial conditions the superposition of precession and nutation can produce for the upper end of the axis, instead of the cycloid-like, wavy (sine-like) or loopy trajectories. A wavy trajectory fully straightens and transforms into a circle (see figure 2) only if the initial velocity of the axis is exactly equal to the velocity characteristic of regular precession.

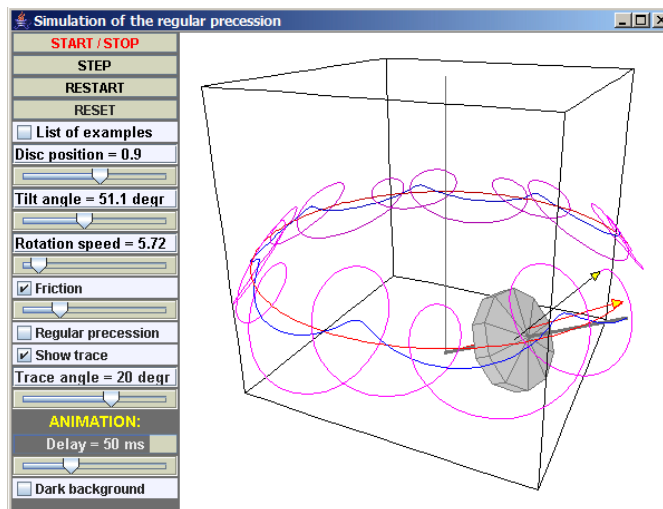
When the cone of nutation is narrow, torque-induced precession is called pseudo-regular. For fast-spinning gyroscopes used in technical applications, pseudo-regular precession is almost indistinguishable from the regular. In these cases nutation reveals itself as hardly noticeable very fast shivering of the gyroscope axis. Moreover, this fast nutation damps out rapidly by virtue of friction, and pseudo-regular precession transforms into a regular precession.

The simulation program allows us to observe the damping out of nutation due to friction. Figure 7 shows how the ‘beaks’ of the axis trajectory smooth over and the curve tends gradually to the circle corresponding to the steady forced precession. This circle (shown in figure 7 by the red line) is generated by the end point of vector  $L_1$ . After nutation fades out completely, the axis assumes the direction of vector  $L_1$  (of the angular momentum due to spin).

Figure 8 gives an impression about the trajectory traced by some point displaced from the axis of the gyroscope during the transition of pseudo-regular precession to steady regular precession. The looped curve is generated by the spike of an arrow fixed to the gyroscope (plunged into its body) and making an angle of  $20^\circ$  with its axis. This curve and the trajectory of the end point of the axis both lie on the same spherical surface whose centre is located at the pivot. When the gyroscope undergoes a regular precession, all the loops of this curve are equal (see figure 2). If the precession is accompanied by nutation, the axis wobbles about a middle position corresponding to the cone of steady precession. Due to this



**Figure 7.** Damping out of nutation due to friction, and smoothing of the axis trajectory during transition to steady precession.



**Figure 8.** Trajectories traced by the axis and by some point displaced from the axis of the gyroscope during the transition of pseudo-regular precession to steady regular precession.

wobbling of the axis, the trajectory of the arrow spike is more complicated—its loops are no more equal.

### Concluding remarks

In this paper we have reconsidered an important old problem of the torque-induced rotation of a spinning symmetrical top. The origin of nutation that accompanies forced precession is explained in detail, with an accent on its relationship to the torque-free precession of

a symmetrical body. Our treatment is appropriate for teaching introductory mechanics and general physics to undergraduate students. The approach suggested is free from the difficulties inherent in a traditional treatment of the problem. A simulation program (Java-applet) [7] visualizes the investigated motion and is intended to help students in developing physical intuition about the phenomenon.

## References

- [1] Goldstein H 1980 *Classical Mechanics* 2nd edn (Reading, MA: Addison-Wesley)
- [2] Goldstein H, Poole Ch P and Safko J L 2002 *Classical Mechanics* 3rd edn (Reading, MA: Addison-Wesley)
- [3] Chabay R and Sherwood B 2002 *Matter and Interactions. V I Modern Mechanics* (New York: Wiley)
- [4] Machlup S 1988 *Physics* (New York: Wiley)
- [5] Kittel Ch, Knight W D and Ruderman M A 1965–1971 *Mechanics. Berkeley Physics Course* (New York: McGraw-Hill)
- [6] Landau L D and Lifschitz E M 1976 *Mechanics* (New York: Pergamon)
- [7] Butikov E 2006 *Forced Precession of a Gyroscope* <http://www.ifmo.ru/butikov/Applets/Gyroscope.html>
- [8] Butikov E 2006 Inertial rotation of a rigid body *Eur. J. Phys.* **27** 913–22
- [9] Butikov E 2006 *Free Rotation of an Axially Symmetrical Body* <http://www.ifmo.ru/butikov/Applets/Precession.html>