The velocity hodograph for an arbitrary Keplerian motion

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Abstract.   An interesting, useful, and simple, but not widely known property of Keplerian motion relating to the circular shape of the orbit in velocity space is discussed in this paper. The property is illustrated by a computer simulation program. A simple dynamical derivation of the circular shape of the velocity hodograph is suggested.
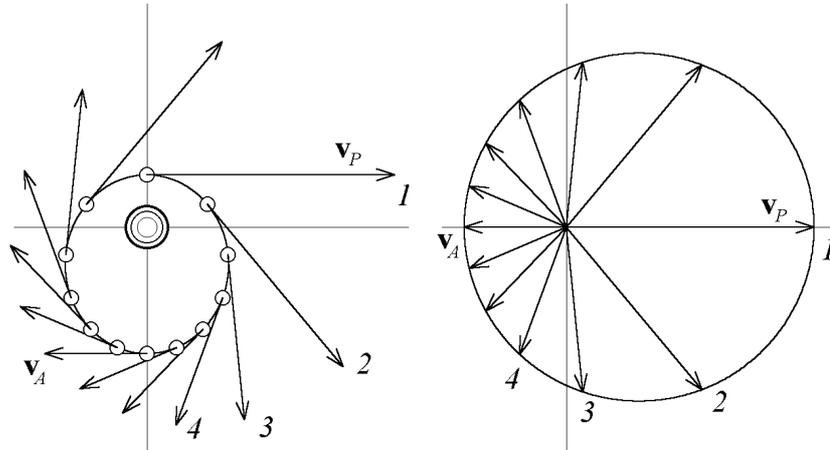
1. Circular hodograph of the velocity vector for closed and open orbits

One of the most interesting aspects of Keplerian motion (the motion along conic sections governed solely by a central force whose magnitude is inversely proportional to the square of the distance from the force centre) concerns the shape of its trajectory in velocity space (or momentum space).

The velocity vector of a moving body at any moment is directed tangentially to the spatial trajectory, so that in curvilinear motion the direction of the velocity vector changes continuously. We obtain the trajectory of motion in *velocity space* as follows. For each point on the spatial trajectory, we draw the corresponding velocity vector so that its tail lies at the origin of velocity space and its direction is parallel to the tangent to the spatial trajectory at the point in question. During the curvilinear non-uniform motion of the body, the direction and magnitude of this vector change. The tip of this varying velocity vector generates a curve in velocity space. The now-customary name of 'hodograph' was given to this curve by Hamilton in 1846.

For a circular orbit, the magnitude of the velocity is constant and so the variation of the velocity vector is reduced to a uniform rotation about the origin of velocity space. It is evident that the hodograph of the velocity vector for the circular Keplerian motion is itself a circle whose centre is located at the origin of velocity space. The radius of this circle equals the constant magnitude of the circular velocity.

As a planet or a satellite moves along a closed elliptical orbit or along an open parabolic or hyperbolic trajectory, rotation of the velocity vector is non-uniform, and both the direction and magnitude of the vector change. However, these variations occur in such a way that the end of the velocity vector in this case also generates a circle (or an arc of a circle) in velocity space but whose centre is not at the origin. In other words, the hodograph of the velocity vector for an arbitrary Keplerian motion is a circle. This interesting property is ignored in almost all the numerous textbooks on mechanics and general physics that treat the orbital motion. For closed orbits, the property is briefly discussed in an optional supplement in [1], and is used in [2] for a

Figure 1. Keplerian orbit of a satellite and the velocity vectors in space (left), and hodograph of the velocity vector in velocity space (right). Identical numbers on the left- and right-hand sides refer to the same instants of time.

geometrical derivation of Kepler's first law. In addition, an appropriate task is assigned in [3] for numerical solution.
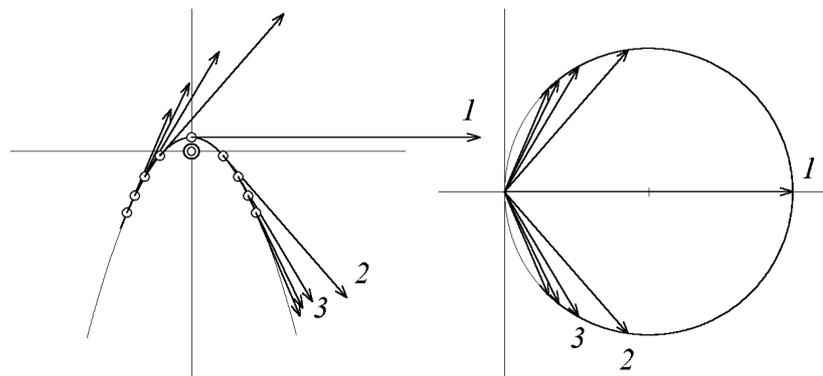
The aim of this paper is to draw attention to this interesting problem and suggest a simple analytic proof based on Newton's laws of motion. But first we illustrate this property with the help of the simulation program included in the educational software package [4], developed recently by the author. The left-hand side of figure 1 shows the spatial trajectory of a satellite or a planet that moves under the inverse-square central force. In equal time intervals, the velocity vectors are fixed as tangents to the spatial trajectory. On the right-hand side the same fixed vectors are shown in velocity space. We can see clearly that during the motion of the satellite along the elliptical orbit, the end of the varying velocity vector generates a circle in velocity space.

The lower semicircle of the hodograph (starting at point 1 and moving clockwise) corresponds to the right-hand half of the elliptical orbit, during which the satellite moves clockwise from perigee towards the apogee with a decreasing speed. The other (upper in figure 1) semicircle of the hodograph corresponds to the second part of the orbit, during which the satellite moves clockwise from apogee to perigee and its speed increases.

For an elliptical orbit, the diameter of this circular hodograph equals the sum of the magnitudes of the velocities v_P and v_A at perigee and apogee, respectively. At these points of an elliptical orbit the velocity vectors are oppositely directed. The centre of this circular hodograph corresponding to the elliptic motion is displaced from the origin of velocity space in the direction of the velocity vector at perigee, v_P, through the distance $(v_P - v_A)/2$ equal to half the difference of the velocity magnitudes at perigee and apogee of the orbit (see figure 1).

At the end of a minor axis of the elliptical orbit, the velocity vector v is parallel to the major axis. Hence v is oriented perpendicularly to the diameter of the circular hodograph. Joining the tip of v to the tips of v_P and v_A in velocity space, we obtained the right-angled triangle inscribed in the hodograph. In this triangle v is the height dropped to the hypotenuse. Therefore, the speed v at the end of the minor axis equals the geometric mean value of the speeds at perigee and apogee of the orbit: $v = \sqrt{v_P v_A}$.

In figure 1 the positions of the satellite in the orbit and the corresponding velocity vectors are fixed at equal time intervals, and we can easily judge variations in the angular velocity of rotation of the velocity vector as it traces the circle. Comparing the angles between sequential positions of the velocity vector, we see that near position 1, as the satellite passes through
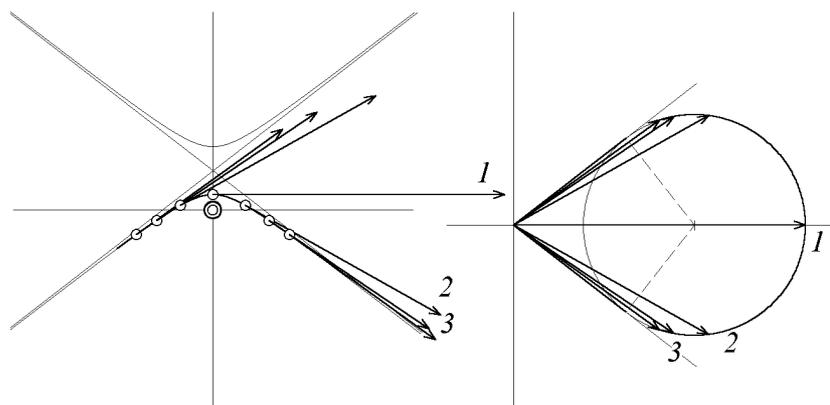
Figure 2. Parabolic trajectory of a body in a central gravitational field and the velocity vectors in space (left), and hodograph of the velocity vector in velocity space (right). Identical numbers on the left- and right-hand sides refer to the same instants of time.

perigee, the velocity vector rotates rapidly. Its rotation slows down as the satellite passes through apogee of the orbit.

An open parabolic trajectory for Keplerian motion (the total energy for which is zero) can be considered the limiting case of motion along a strongly elongated elliptical orbit whose apogee recedes to infinity (figure 2). In this case, the velocity of the satellite at apogee approaches zero. The hodograph is a closed circle whose diameter equals the velocity of the body at the vertex of the parabola (the point on the trajectory nearest to the centre of force). The circular hodograph corresponding to the parabolic Keplerian motion passes through the origin of velocity space. The origin is reached when the body recedes to infinity. This motion is of infinite duration.

For an infinite hyperbolic motion in an inverse-square central gravitational field, the hodograph of the velocity vector is also circular (figure 3). In this case, the radius of the velocity hodograph is smaller than the maximal velocity v_P of the body at the point nearest to the centre of force. The origin of velocity space is outside the circle.

In hyperbolic motion, the body approaches the centre of force from infinity, where its

Figure 3. Hyperbolic trajectory of a body in a central gravitational field and the velocity vectors in space (left), and hodograph of the velocity vector in velocity space (right). Identical numbers on the left- and right-hand sides refer to the same instants of time.

velocity is non-zero and is directed along one of the asymptotes of the hyperbola. In the hodograph, the velocity at infinity is tangential to the circle. In the course of motion the velocity vector generates a part of the circle away from the origin. Starting from the point of tangency, the magnitude of the vector gradually increases and reaches its maximal value when the vector extends from the origin of velocity space to the furthest point of the circle. This vector is the velocity of the body when the body passes through the vertex of the hyperbolic trajectory (the point nearest to the force centre).

The velocity vector then gradually shortens, its end moving further along the circle. Eventually the end of the velocity vector reaches the other point of tangency where the velocity assumes its initial magnitude and the body is infinitely remote. This second tangent to the hodograph is directed parallel to the other asymptote of the hyperbolic trajectory, along which the body recedes.

2. A simple dynamical derivation of the circular shape of the velocity hodograph

According to Goldstein [5], the circular shape of the hodograph for the elliptical orbits was first communicated in 1846 by Hamilton, whose investigation was based on the use of the invariant Laplace or Runge–Lenz vector. This vector can be referred to as a dynamical invariant since it depends on the particular force law (specifically, the inverse-square law), while such invariants as the angular momentum and energy can be called geometrical, since they depend on general symmetry properties of space and time. Hamilton proves that the Laplace vector is constant as a preliminary to demonstrating that the hodograph is circular. Certainly, the invariant nature of the Laplace vector and the circular shape of the velocity hodograph are closely related.

Nevertheless, it is possible to give a simple, but rigorous proof of the circular hodograph theorem without referring to the Laplace vector. The theorem can be proved geometrically, in the manner favoured by Newton. The proof is based directly on Newton's laws of motion applied to a particle that moves under the central gravitational force $F(r) = GmM/r^2$.

According to Newton's law of motion, the effect of force is to modify the momentum of the body, that is, its velocity, rather than its position. The vector dv/dt of the acceleration produced by the force of gravity is always directed towards the centre of force, and its magnitude is inversely proportional to the square of distance r:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -GM\frac{1}{r^2}\frac{r}{r}. \tag{1}$$

Next we eliminate the variable $1/r^2$ from this equation using the law of conservation of angular momentum:

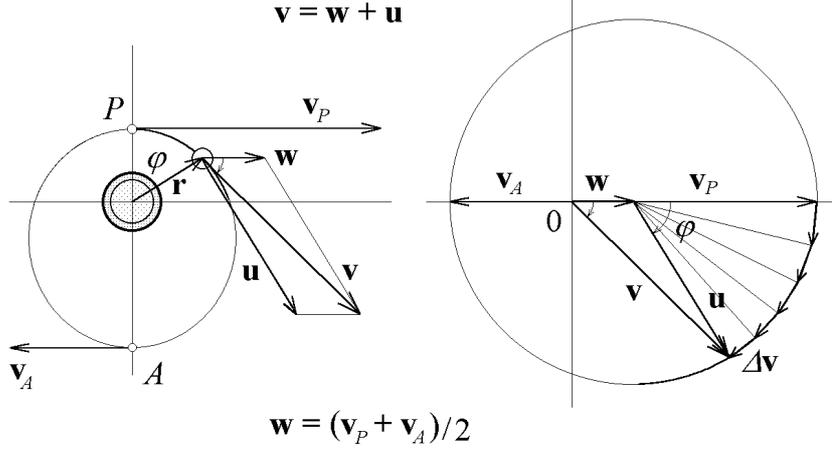$$L = m|r \times v| = mrv_\perp = mr^2\dot{\varphi}. \tag{2}$$

Here $v_\perp = r\dot{\varphi}$ is the transverse component of the velocity of the particle (the component orthogonal to the radius vector), and $\dot{\varphi} = \mathrm{d}\varphi/\mathrm{d}t$ is the angular velocity. Since the angular momentum of the particle remains constant during the motion in a central force field, for any point of the trajectory the square of the distance r from the origin can be expressed with the help of equation (2) in terms of the angular velocity $\dot{\varphi}$ and a constant value of L:

$$\frac{1}{r^2} = \frac{m}{L}\dot{\varphi} = \frac{m}{L}\frac{\mathrm{d}\varphi}{\mathrm{d}t}. \tag{3}$$

Substituting $1/r^2$ given by equation (3) into equation (1), we find that the increment in the velocity vector during the time interval Δt is given by the infinitesimal vector Δv whose direction is antiparallel to r and whose magnitude is proportional to the angle $\Delta\varphi$, through which the radius vector of the particle rotates during Δt:

$$|\Delta v| = \frac{GMm}{L}\Delta\varphi. \tag{4}$$

We note that this proportionality between $|\Delta v|$ and $\Delta\varphi$ holds only for the motion in a central field whose strength is proportional to $1/r^2$. Hence the circular form of the velocity hodograph (the property that we are going to prove) is an inherent property of inverse-square central fields.

$$\mathbf{v} = \mathbf{w} + \mathbf{u}$$

$$\mathbf{w} = (\mathbf{v}_P + \mathbf{v}_A)/2$$

Figure 4. Keplerian orbit of a satellite and the velocity vector in space (left), and hodograph of the velocity vector in velocity space (right).

A geometric interpretation of equation (4) is shown in figure 4. Each time the spatial radius vector r of the orbiting particle turns through an infinitesimal angle $\Delta\varphi$, the velocity vector v is incremented by Δv, whose magnitude $|\Delta v|$ is proportional to $\Delta\varphi$. This implies that the elementary vectors Δv in velocity space lie along a circle, that is, the polygonal line formed by sequential Δv vectors is circumscribed by a circle (figure 4). The radius u of this circle, according to equation (4), equals GMm/L. At each moment the radius vector u of the hodograph is perpendicular to Δv and, hence, to the spatial radius vector r. In other words, the (non-uniform) rotation of the position vector r in space (left-hand side of figure 4) and the corresponding rotation of vector u in velocity space (right-hand side of figure 4) always occur $90°$ out of phase.

Thus, equation (4) proves that the hodograph of the velocity vector is a circle. The constant value $u = GMm/L$ of the coefficient of proportionality between $|\Delta v|$ and $\Delta\varphi$ in equation (4) gives the radius of this circular hodograph of the velocity vector.

It is convenient to express radius u of the velocity hodograph in terms of velocity v_P at perigee and the circular velocity v_c for the perigee distance r_P. Since $v_c = \sqrt{GMm/r_P}$ and $L = mr_Pv_P$, we obtain the expression:

$$u = \frac{v_c^2}{v_P}. \tag{5}$$

If the satellite is launched into a circular orbit ($v_P = v_c$), equation (5) yields $u = v_c$. In this trivial case the radius of the hodograph clearly equals the circular velocity. For an elliptical orbit, it is possible to express the radius $u = GMm/L$ of the hodograph in terms of velocities v_P and v_A at the perigee and apogee, respectively. We can use the laws of the conservation of energy and angular momentum for this purpose. Equating the values of the total energy at these points r_P and r_A, we write:

$$\frac{m}{2}v_P^2 - G\frac{Mm}{r_P} = \frac{m}{2}v_A^2 - G\frac{Mm}{r_A}. \tag{6}$$

We next solve this equation for GMm, and substitute into $u = GMm/L$ the expression obtained, together with the (constant) angular momentum L, calculated, say, for the perigee:

$L = mr_P v_P$. The ratio of distances r_P/r_A can be eliminated with the help of the relation $r_P v_P = r_A v_A$. (The values of the angular momentum are equal at the perigee and apogee). Finally, we obtain the following expression for the radius u of the velocity hodograph:

$$u = \frac{1}{2}(v_P + v_A). \tag{7}$$

By virtue of this property, we can represent the velocity vector v for any point of an elliptical orbit as the vector sum of the following two vectors w and u (see figure 4). One term of the sum is the constant vector $w = (v_P + v_A)/2$ of magnitude $(v_P - v_A)/2$, directed along the vector v_P of the velocity at perigee. This vector w extends from the origin of velocity space to the centre of the circular hodograph. The second term is a vector u of constant magnitude $u = (v_P + v_A)/2$, whose direction is always perpendicular to the spatial radius vector r of the orbiting body.

This representation of the velocity vector as the sum of two vectors ($v = w + u$) is useful for solving certain problems concerning Keplerian motion.

The derivation of equation (4) that gives the proof of the circular form of the velocity hodograph, as well as the geometric interpretation of equation (4) discussed above, are based on the inverse-square dependence of the central force on distance. The assumption about the closed trajectory is used only in the calculation of the hodograph radius u in terms of v_P and v_A. Consequently, the circular form of the velocity hodograph is characteristic not only of closed (circular and elliptical) orbits, but also of open (parabolic and hyperbolic) trajectories of motion in a central field whose force is inversely proportional to the square of the distance from the centre. For the parabolic trajectory the two terms of the sum $u + w$ have equal magnitudes: $u = w = v_P/2$. For a hyperbolic motion, the diameter $2u$ of the circular hodograph is smaller than the velocity v_P at the vertex of the hyperbola (see figure 3). In this case $u < w$ and the origin of velocity space is located outside the circular hodograph.

References

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