Orbital Maneuvers and Space Rendezvous

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Abstract.
Several possibilities to launch a space vehicle from the orbital station which can be useful in designing a trajectory for a specific mission are discussed and compared. The relative motion of orbiting bodies is investigated on examples of spacecraft rendezvous with the space station that stays in a circular orbit around the Earth. An elementary approach is illustrated by a simulation computer program and supported by a mathematical treatment based on fundamental laws of physics and conservation laws. Material is appropriate for engineers and other folks involved in space exploration, undergraduate and graduate students studying classical physics and orbital mechanics.

Key words: Keplerian orbits, conservation laws, period of revolution, space navigation, impulsive maneuvers, characteristic velocity, Hohmann's transitions, soft docking.

1. Introduction: Designing Space Flights and Orbital Maneuvers

One of the first problems that designers of space missions faced was figuring out how to go from one orbit to another. Many important problems in astrodynamics are associated with modifying the orbit of a satellite or a spacecraft in order to produce a particular trajectory for an intended space mission. The orbit can be modified by applying a brief impulse to the craft. In particular, the velocity of the craft can be changed by the thrust of a rocket engine that is so oriented and of such duration as to produce the desired result. The maneuver should be executed at a proper instant by the astronauts of the spacecraft or by a system of remote control.

In unusual conditions of the orbital flight, navigation is quite different from what we are used to on the Earth's surface (or even in the air or under the water), and our intuition fails us. The orbital maneuvers are not as simple as driving a car or a motor boat from one point to another. Even flying a spacecraft is very different than flying a plane.

If two satellites are brought together but have a (small) nonzero relative velocity, they will drift apart non-rectilinearly. Because a spacecraft is always in the gravitational field of some central body (such as Earth or the Sun), it has to follow orbital-motion laws in getting from one place to another. To properly understand the rendezvous of spacecraft, it is essential to understand the laws that govern the passive motion in a central field of gravity.

When the rocket engine is very powerful and operates for a very short time (so short that the spacecraft covers only a very small part of its orbit during the thrust), the change in the orbital velocity of the spacecraft is essentially instantaneous. Most propulsion systems operate for only a short time compared to the orbital period, so that we can treat the maneuver as an impulsive change in velocity while the position remains unchanged. In this paper it is assumed that the change in velocity occurs instantly. After such a maneuver the spacecraft continues its passive orbital motion along a new orbit. The parameters that characterize the new orbit depend on the initial conditions implied by momentary values of the radius vector and the velocity vector of the spacecraft at the end of the applied impulse.

The aims of orbital maneuvers may be varied. For example, we may plan a transition of the vehicle undocked from the orbital station into a higher circular orbit in order to remain in it for some time, eventually returning to the station and soft docking to it. Or we may wish to design a transition of the landing module to a descending elliptical orbit that grazes the Earth's surface (the dense strata of the atmosphere) in order to return to the Earth from the initial circular orbit. We may want to launch from the orbital station an automatic space probe that will explore the surface of the planet from a low orbit, or, on the other hand, to send a probe far from the Earth to investigate the interplanetary space. The orbit of the space probe must be designed to make possible its return to the station after the mission is over. Several types of missions require a spacecraft to meet or rendezvous with another one, meaning one spacecraft must arrive in the same place at the same time as a second one. A rendezvous also takes place each time a spacecraft brings crew members or supplies to an orbiting space station.

To plan such space flights, we must solve various problems related to the design of suitable transitional orbits. We must decide how many instant maneuvers are necessary to reach the goal. To make each transition of the space vehicle into a desired orbit, we must calculate beforehand the magnitude and direction of the required additional velocity (the characteristic velocity), as well as the time at which this velocity is to be imparted to the space vehicle. As a rule, the solution of the problem is not unique.

The complexity of the problem arises from the expectation that we choose an optimal maneuver from many possibilities. The problem of optimization may include various requirements and restrictions concerning admissible maneuvers. For example, there may be a requirement of minimal expenditures of the rocket fuel, with an additional condition that possible errors in the navigation and control (in particular, errors in the time of executing the maneuver and inevitable errors in direction or magnitude of the additional velocity) do not cause inadmissible deviations of the actual orbit from the predicted (calculated) one.

Various problems related to orbital mechanics and astrodynamics are discussed in a lot of texts and papers (see, for example, [1]–[7]). Many useful references can be found on the web [8]. In the present paper we discuss orbital maneuvers needed for safe landing and for rendezvous of spacecraft. To keep things simple, we assume that the initial and final orbits are in the same plane. Such maneuvers are often used to move spacecraft from their initial parking orbits to their final mission orbits. It is also assumed in this paper that originally the active spacecraft is docked with a permanent space station that orbits the Earth (or some other planet) in a circle. The additional velocity (sometimes called the characteristic velocity) needed to transfer the spacecraft to a desired new orbit is imparted to the spacecraft by the on-board rocket engine after undocking.

Examples and basic principles of orbital maneuvers are described in the body of this paper without heavy mathematics, on a qualitative level accessible to a wide readership. A mathematical justification can be found in the Appendices. We pay special attention to the motion of the undocked spacecraft relative to the orbital station. The motion of the spacecraft both relative to the Earth and as observed by the astronauts of the station is simultaneously illustrated by computer simulations [9] (see in the web http://butikov.faculty.ifmo.ru/ (section Downloads, program “Planets and Satellites”). The simulations reveal many extraordinary features that are hard to reconcile with common sense and our everyday experience.

2. Way Back from Space to the Earth

As an example of active maneuvers of a spacecraft staying originally in a low circular orbit around a planet, we consider the problem of transition of a landing module to a descending trajectory. For a safe return to the Earth, the landing module must enter the dense strata of the atmosphere at a very small angle with the horizon. A steep descend is dangerous because of the rapid heating of the spacecraft in the atmosphere. The thermal shield of the landing module must satisfy very stringent demands. For a manned spacecraft, large decelerations caused by the air drag at a steep descend are inadmissible mainly because of the dangerous increase in the pseudo weight of the space travelers. All this means that the planned passive descending trajectory must just graze the upper atmosphere.

Next we shall consider and compare two possible ways to transfer the landing module into a suitable descending trajectory.

(i) After the landing module is undocked from the orbital station, it is given an additional velocity directed opposite to the initial orbital velocity.

(ii) The additional velocity of the landing module is directed downward (along the local vertical line).

In all cases, any additional velocity transfers the space vehicle from the initial circular orbit to an elliptical orbit. One of the foci of the ellipse is located, in accordance with Kepler’s first law, at the center of the Earth.

In the first case, a brief operation of the rocket engine changes only the magnitude of the orbital velocity, preserving its direction. Therefore, at the point where the rocket engine operates (point A in the left panel of Figure 1), the descending semielliptical orbit has a common tangent with the original circular orbit. This point A is the apogee of the elliptical orbit of the landing module. Its perigee is located at the opposite end P of the major axis. This axis passes through A and the center of the Earth E. The remote (from the apogee A) focus of the ellipse is located At this point E. The second focus of this ellipse is located at point F in Figure 1.

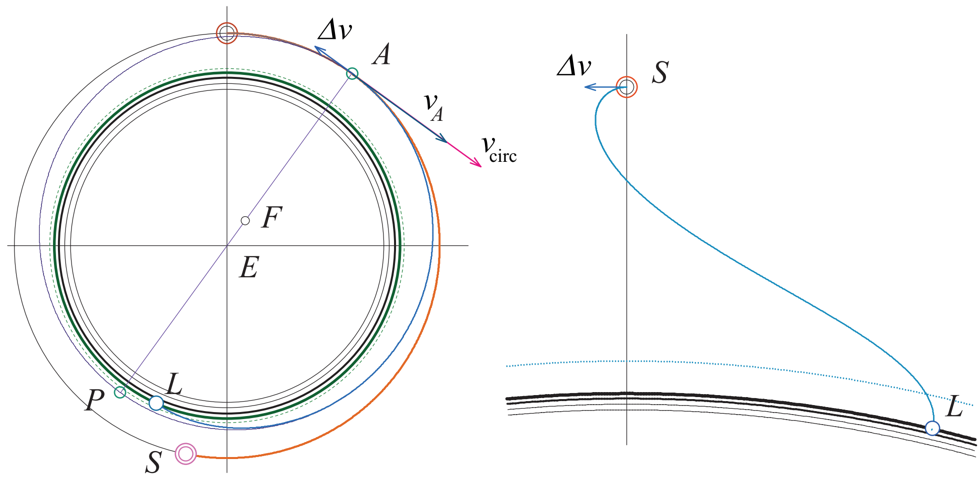

Figure 1. Descending elliptical trajectory of the landing module after a backward impulse is applied at point A (left), and the descent of the module as it appears to the astronauts on the orbital station S (right). The thin dotted line shows a conventional upper boundary of the atmosphere.

The additional velocity Δv must be chosen from the requirement that at point P the ellipse must just graze the surface of the planet. More exactly, the ellipse must graze the upper strata of the atmosphere in order the landing module enter the atmosphere at a very small angle before reaching this point P of the descending orbit. For this method of landing, the angular distance between the initial point and the point of landing is approximately 180°.

The simulation of landing presented in Figure 1 uses one of the programs of the software package [9]. It is based on numerical integration of the equation of motion of the landing module. The simulation shows that the landing module actually moves along the theoretically predicted ellipse during almost all first half of revolution around the Earth. But near the point P its trajectory deviates downward due to the air resistance, which is taken into account in the simulation. The landing module reaches the ground at point L. To make the influence of the air drag more noticeable, the height and density of the atmosphere is exaggerated in the simulation. A conventional boundary of the atmosphere (whose density reduces exponentially with the height over the surface) is shown by the dotted line in Figure 1. At the moment of landing the station passes through point S of its circular orbit.

The right-hand panel of Figure 1 shows the descent of the module in the frame of reference associated with the orbital station. At first the landing module actually moves relative to the station backwards, that is, in the direction of the additional velocity. However, very soon its relative velocity turns downward and reverses. Gradually descending, the module moves forward and overtakes the station, leaving it far behind. We note that near the Earth the trajectory steeply bends towards the surface. This is caused by the increasing air resistance. Due to the same reason, the point of landing occurs before the module reaches the perigee P of the ellipse.

The additional (characteristic) velocity Δv necessary for the transition from the circular orbit to this elliptical trajectory can be calculated from the conservation laws of energy and angular momentum. Details of the calculation can be found in Appendix I. We present here only the resulting formula:

$$\Delta v = v_{\text{circ}}\left(1 - \sqrt{\frac{2}{1 + r_0/R}}\right). \tag{1}$$

Here $v_{\text{circ}} = \sqrt{GM/r_0} = \sqrt{gR^2/r_0}$ is the circular velocity of the space station, G is the gravitational constant, M is the mass of the planet, g is the acceleration of free fall, r_0 is the radius of the circular orbit, and R is the Earth's radius (more exactly, radius of the Earth together with the atmosphere).

In the case of a low circular orbit, whose height $h = r_0 - R$ over the Earth is small compared to the Earth's radius ($h \ll R$), the exact equation, Eq. (1), can be replaced by an approximate expression:

$$\Delta v \approx v_{\text{circ}}\frac{h}{4R}. \tag{2}$$

For example, if the height h of the circular orbit equals $0.2\,R \approx 1270$ km, the additional velocity Δv, according to Eq. (2), must be about 5% of the circular velocity. (The calculation on the basis of Eq. (1) with $r = R + h = 1.2\,R$ gives a more exact value of 4.65%).

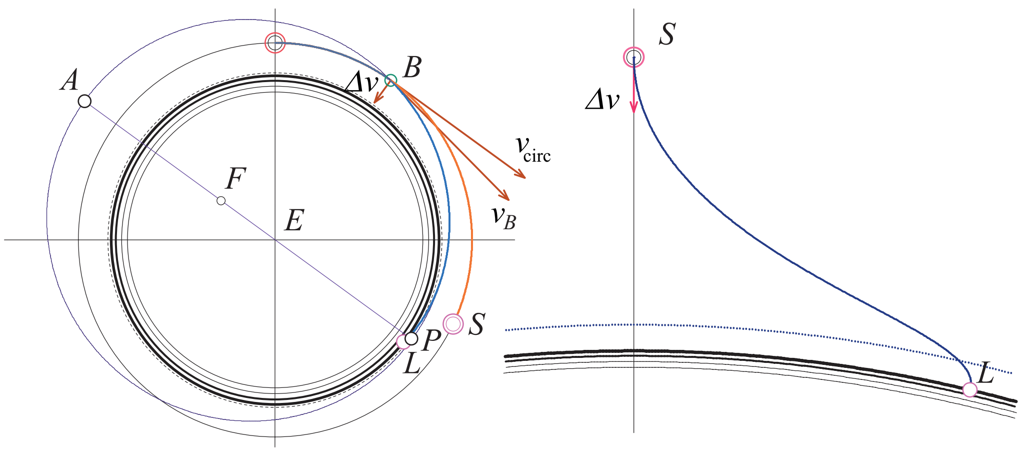

Figure 2. Descending elliptical trajectory of the landing module after given a downward additional velocity at point B (left), and the descent of the module as it appears to the astronauts on the orbital station S (right).

If the additional velocity imparted to the space vehicle at point B of the initial circular orbit (Figure 2) is directed radially (transverse to the orbital velocity), both the magnitude and direction of the velocity change. Therefore, the new elliptical orbit intersects the original circular one at this point B. For a soft landing, the new elliptical trajectory of the descent must also graze the Earth (the upper atmosphere) at the perigee P of the ellipse. Using the laws of energy and angular momentum conservation and requiring that the perigee distance

r_P be equal to the Earth's radius R, we can find (see Appendix I) that the necessary additional velocity Δv for this method of landing is given by

$$\Delta v = v_{\rm circ}\,\frac{h}{R}. \tag{3}$$

Here $v_{\rm circ}$ is the circular velocity for the original orbit, h is its height above the surface (above the atmosphere), and R is the Earth's radius (including the atmosphere). Comparing this expression with Eq. (2), we see that for this method of transition to the landing trajectory, the required additional velocity is approximately four times greater than that for the first method. For example, it must equal 20% of the circular velocity, if the height h of the circular orbit is $0.2\,R$. The angular distance between the starting point B and the landing point L (see Figure 2) for this method equals 90° (a quarter of the revolution), in contrast to the first method, for which the angular distance between the point of transition from the circular orbit to the descending trajectory and the landing point is twice as large (half a revolution).

Figure 2 also shows position S of the orbital station at the moment of landing. We can see that the station is above and some distance behind the landing module since for the moment of landing the station has not completed a quarter of its revolution beyond the initial point B.

The right-hand side of Figure 2 shows the landing trajectory in the frame of reference associated with the orbital station. At first the astronauts on the station see that the landing module really moves downward, in the direction of the additional velocity imparted by the on-board rocket engine. However, soon the trajectory bends forward, in the direction of the orbital motion of the station. The landing module in its way towards the ground moves forward and overtakes the station, leaving it in its orbital motion far behind.

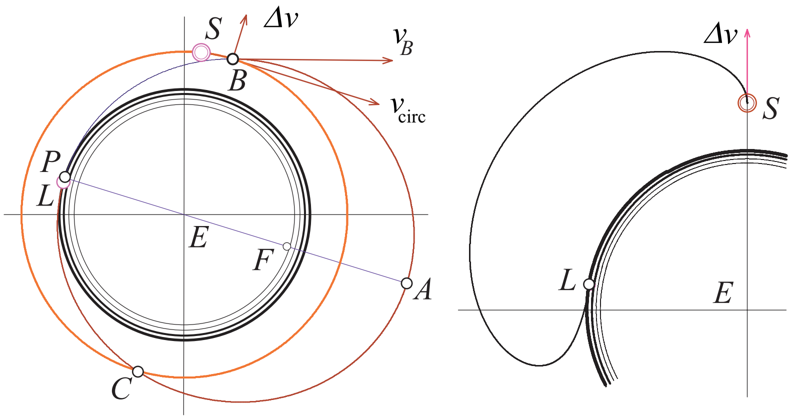

Figure 3. Elliptical trajectory of the landing module after acquiring an upward additional velocity at point B (left), and the trajectory of the module as it appears to the astronauts on the orbital station S (right).

Strange as it may seem, we can transfer the space vehicle to a landing trajectory by a transverse impulse directed vertically upward as well as downward (Figure 3). In this case, starting from the point B of transition to the elliptical orbit, the landing module first rises higher above the Earth. Only after it passes through the apogee A of the orbit does it begin to descend toward point P (the perigee of the orbit), at which it enters the atmosphere. The angular distance between the starting and the landing points (B and L, respectively) in this case equals approximately $270°$, that is, about three quarters of a revolution. During this time, the orbital station covers almost a whole revolution, and at the moment the vehicle lands at point L, the station S is far beyond the landing point.

The trajectory of the landing module as it is seen by the astronauts in the orbital station is shown in the right-hand panel of Figure 3. The module first moves upward, in the direction of the additional velocity, but soon turns backward. Its relative motion becomes retrograde, and the landing module lags behind the station. After circling more than a quarter of the globe in the retrograde direction, the module's motion reverses direction. The module then descends, approaching the Earth's surface tangentially.

For an elliptical orbit that is to graze the Earth, the magnitude of the additional velocity must be the same for both the downward and upward directions of the impulse. We can easily see this point either from the laws of the conservation of energy and angular momentum (the corresponding equations are the same for both cases, see Appendix I), or from considerations based on the symmetry between the two cases: for if the goal is to land the module near some point P of the Earth's surface (Figure 2), we must make a transition from the initial circular orbit to an elliptical orbit for which point P is the perigee. The orbits intersect at two points B and C. The transition is possible either at B using an upward impulse, or at a symmetrical point C using a downward impulse of equal magnitude.

The method of descending from a circular orbit with the help of a backward impulse requires the absolutely minimal amount of rocket fuel. However, this method is very sensitive to even small variations in the value (and direction) of the additional velocity. In the ideal situation, if the additional velocity has exactly backward direction and the required value given by Eq. (1), the point of landing L is near the perigee P of the ellipse (see Figure 1). During the descent, the landing module covers about one half of the ellipse (from A to P) while the station covers a little less than half its circular orbit. At the moment of landing, the station is above and a little behind the module (point S in Figure1).

The sensitivity of this method to variations in the additional velocity means that if the actual magnitude of the additional velocity is slightly greater than the required value, the point of landing L moves considerably from the idealized perigee (point P) towards the starting point A. And if the velocity Δv is smaller than required, the perigee of the elliptical orbit occurs above the dense strata of the atmosphere, and the space vehicle may stay in the orbit for several loops more. Because there is considerable air resistance near perigee, the apogee of the orbit gradually descends after each revolution. The orbit approaches a low circle. Eventually the space vehicle enters the dense atmosphere and lands. However, it is almost impossible to predict when and where this landing occurs. To avoid such complications, in practice of orbital flights the additional velocity that transfers the landing module to a descending trajectory is chosen usually to have also a downward transverse component.

3. Transitions between Orbits and Interplanetary Flights

Next we discuss the space maneuvers that can transfer a space vehicle from one circular orbit to another.

Suppose we need to launch a space vehicle from the orbital station into a circular orbit whose radius is different from that of the space station. After remaining in this new orbit for a while, the space vehicle is to return to the orbital station and dock to it. What maneuvers must be planned to execute this operation? What jet impulses are required for optimal maneuvers? What characteristic velocities must the rocket engine provide?

Designing such transitions between different circular orbits can be related also to interplanetary space journeys. The orbits of the planets are almost circular, and to a first approximation they lie in the same plane. In a sense, planets are stations orbiting the sun. Sending a space vehicle from one planetary orbit to another differs from the problem suggested above only in that the planets (unlike actual stations) exert a significant gravitational pull on the space vehicle. But since masses of the planets are small compared to the mass of the sun, the gravitational field of a planet is effective only in a relatively small sphere centered at the planet. Outside this sphere of gravitational action of the planet the motion of a space vehicle (relative to the heliocentric reference frame) is essentially a Keplerian motion governed by the sun. In this sense the problem of interplanetary flights is quite similar to the problem to be discussed here. The only difference is that in the case of interplanetary flights the additional velocity needed to simulate a maneuver on the computer should be treated as the velocity with which the space vehicle leaves the sphere of gravitational action rather than the surface of the planet.

Because fuel is critical for all orbital maneuvers, we look first of all at the most fuel-efficient method: the so-called Hohmann's transfer. In 1925 a German engineer, Walter Hohmann, suggested a certain way to transfer between orbits. It is amazing that he was thinking about this at those old times, many years before launching artificial satellites became technically possible. This method uses a semielliptical transfer orbit tangent to the initial and final orbits (a semielliptic trajectory that grazes the inner orbit from the outside and the outer orbit from the inside).

As a particular example, we next consider the voyage of a spacecraft from an orbital station that moves around a planet in an inner circular orbit of radius r_0 to an outer circular orbit of radius $2r_0$. After remaining in this new orbit for a while, the spacecraft returns to the orbital station. Figure 4 illustrates the maneuvers. At point P_1 the space vehicle is undocked from the station and the on-board rocket engine imparts to the vehicle an additional velocity Δv_1 in the direction of the orbital motion. In order to acquire an apogee of $2r_0$ for the transitional semielliptic trajectory, the additional velocity Δv_1 must equal $0.1547\, v_{\rm circ}$, where $v_{\rm circ}$ is the orbital velocity of the station. The calculation of the required additional velocity Δv_1 on the basis of the laws of conservation of the energy and angular momentum is given in the Appendix II. When the space vehicle reaches the apogee (point A_1) of the ellipse, a second tangential impulse Δv_2 is required to increase the velocity from v_A to the certain value $v_{\rm c}$, in order to place the space vehicle in the outer circular orbit. For this outer orbit, whose radius

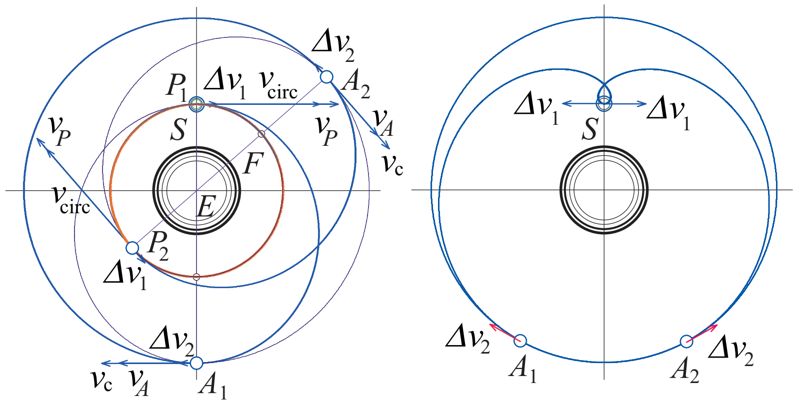

Figure 4. Semielliptic Hohmann's transition of a spacecraft to a higher circular orbit with subsequent return to the orbital station.

is twice the radius r_0 of the station orbit, the circular velocity v_c equals $v_{circ}/\sqrt{2}$, because the circular velocity is inversely proportional to the square root of the orbit's radius.

An additional velocity of the same magnitude Δv_2 but directed opposite to the orbital velocity is required to transfer the space vehicle to a semielliptic trajectory that can bring it back to the station. However, when the orbital station is to be the target, another important consideration is timing: The station must be in the right spot in its orbit at just the moment when the space vehicle arrives. Therefore the instant and the point A_2 (see Figure 4) at which the maneuver is carried out must be chosen properly in order that the space vehicle reach the perigee P_2 simultaneously with the station. To calculate a suitable time, we can use Kepler's third law (see Appendix II for details).

To equalize the velocity of the space vehicle with the velocity v_{circ} of the station as they both meet at point P_2 (see Figure 4), one more rocket impulse (directed opposite to the orbital velocity) is required. It is obvious that now the required additional velocity has the same magnitude Δv_1 as it does for the very first maneuver.

The right-hand panel of Figure 4 illustrates the motion of the space vehicle in the frame of reference associated with the orbital station. At first the vehicle actually moves forward, in the direction of the additional velocity Δv_1, but very soon its velocity relative to the station S turns up and then backward (the relative trajectory makes a small arc near point S). The further motion of the vehicle relative to the station is retrograde. We note that between points A_1 and A_2 the space vehicle covers more than one revolution around the station in its retrograde relative motion, while in the planetocentric motion between the corresponding points A_1 and A_2 (left-hand panel of Figure 4) it covers less than one revolution.

4. Rendezvous in Space: Soft Docking to the Space Station

An orbital station S moves around the Earth in a circular orbit (clockwise in Figure 5). A spacecraft with crew members and supplies is launched to dock to the station, but because of an unexpected delay at the launch, the craft moves into the same circular orbit some distance behind the station. Docking spacecraft poses serious challenges not encountered, say, when connecting two aircraft for a refueling. Like an air travel, space travel works in three dimensions. But unlike the air travel, there is an additional confusion caused by navigating the craft that are free-falling in orbit.

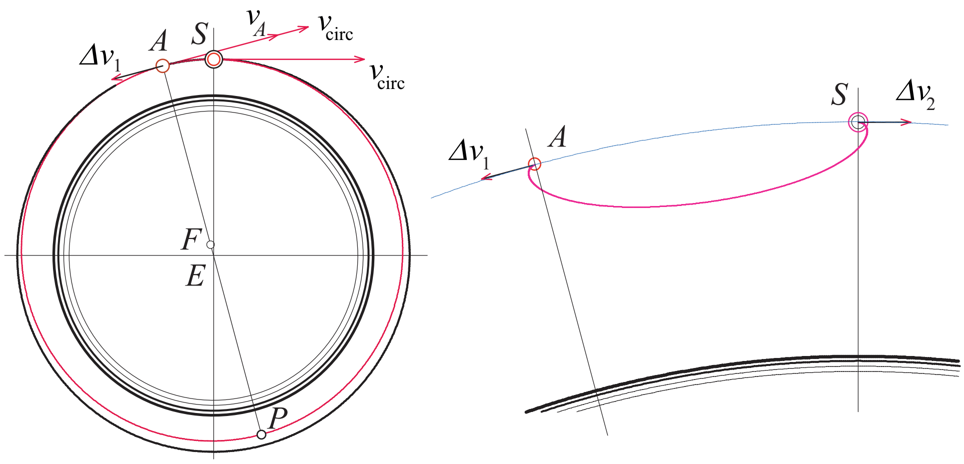

Figure 5. Semielliptic trajectory of a spacecraft needed to reach the orbital station S (left), and how this trajectory looks in the reference frame of the station S (right).

The process of docking two orbiting spacecraft was the focus of Dr. Aldrin's thesis titled "Line-of-Sight Guidance Techniques for Manned Orbital Rendezvous" submitted to MIT. This thesis brought him among colleagues a nickname "Dr. Rendezvous". He was the first PhD in space. The first docking occurred on Gemini-8, in 1966. Soon thereafter, Dr. Aldrin flew aboard Gemini-12 and was able to verify his doctoral work with hands-on experience. Astronauts Neil Armstrong and Buzz Aldrin remain in history as the first human beings to walk on the Moon, in 1969.

Let the distance L between the two spacecraft in the same orbit be small compared to the radius r_0 of the orbit ($L \ll r_0$), see Figure 5. Even in this case simply pointing the active vehicle's nose at the target and thrusting won't do the desired job of the space rendezvous. Dr. Aldrin describes the surprising result of this maneuver as an orbital paradox: "You'll end up in a higher orbit, traveling at a slower speed and watching the second craft fly off into the distance."

The proper technique requires changing the tracking vehicle's orbit to allow the rendezvous target to catch up, and then at the correct moment change to the same orbit as the target with no relative motion between the vehicles.

In order that the spacecraft reach the station, say, after one revolution along the orbit, an additional rocket impulse is required. At first sight it may seem strange, but to reach the station that moves along the same orbit in front of the spacecraft and to catch it up, we should brake. But being armed with understanding how the things work in conditions of an orbital flight, we realize that for the most fuel-efficient method of the rendezvous, a tangential braking impulse is required, which must slow down the spacecraft. This backward thrust transfers the craft to an inner elliptical orbit (see Figure 5) whose apogee A is at the point of thrust, and perigee at the opposite point P.

Point A is the only common point of the circular orbit of the station and the elliptical orbit of the spacecraft. Only at this spatial point the rendezvous is possible. In order the spacecraft arrive at this point simultaneously with the target, the period T of revolution along the ellipse must just equal the lapse of time needed for the station to come from point S (which the station passed through at the moment of thrust) to point A. We must use this condition to calculate the required value of the additional velocity Δv_1 for the maneuver. This can be done by using the conservation laws of energy and angular momentum, and Kepler's third law. Details of the calculation can be found in Appendix III. According to Eq. (23), after the backward thrust the spacecraft must have at point A the following velocity v_A:

$$v_A = v_{\rm circ}\sqrt{2-(T_0/T)^{2/3}}. \tag{4}$$

Hence the required backward additional velocity Δv_1 can be easily calculated:

$$\Delta v_1 = v_{\rm circ} - v_A = v_{\rm circ}\left(1-\sqrt{2-(T_0/T)^{2/3}}\right). \tag{5}$$

This is an exact expression for Δv_1. It can be simplified for the case of small distance L between the tracking vehicle and the target. In this case the elliptical orbit only slightly differs from the circular orbit of the station (see Figure 5), so that we can present the required period as $T = T_0 - \Delta T$ and consider $\Delta T/T_0$ to be a small parameter ($\Delta T/T_0 = L/(2\pi r_0) \ll 1$). This yields for v_A and Δv_1 instead of Eqs. (4) and (5) the following approximate expressions:

$$v_A \approx v_{\rm circ}\left(1-\frac{1}{3}\frac{\Delta T}{T_0}\right), \quad \Delta v_1 \approx v_{\rm circ}\frac{\Delta T}{3T_0}. \tag{6}$$

To illustrate the rendezvous by a computer simulation (see Figure 5), we choose for definiteness the angular distance between the spacecraft and the target station to be 15° (the arc AS in Figure 5), so that $\Delta T/T_0 = L/(2\pi r_0) = 1/24$. The approximate Eq. (6) yields for this case $\Delta v_1 = 0.0139\, v_{\rm circ}$, while the exact Eq. (5) yields $\Delta v_1 = 0.0145\, v_{\rm circ}$ (the value used in the simulation). We see that after one revolution along the ellipse the spacecraft arrives to the apogee A simultaneously with the station.

The right-hand panel of Figure 5 shows the trajectory of the tracking spacecraft in the reference frame associated with the orbital station. At first the craft's motion relative to the station is actually retrograde: it moves backward, in the direction of the additional velocity Δv_1. But very soon the relative velocity turns downward and then forward. The vehicle gradually overtakes the station, rises to the initial altitude, and exactly after a revolution occurs just in front of the station. When the craft reaches the station, one more rocket impulse is required to equalize their velocities for soft docking. The additional velocity Δv_2 required for this maneuver must be of the same magnitude as Δv_1, but must be directed oppositely, in the direction of orbital motion.

If the spacecraft is to approach and dock to the station after two (or n) revolutions along the orbit, the characteristic velocity for the maneuver must be approximately twice (or n times) smaller.

An analytical derivation of the spacecraft trajectory of relative motion in the vicinity of the orbital station on the basis of linearized differential equations of motion is presented in Appendix IV.

5. To the Opposite Side of the Orbit and Back

Next we consider one more example of space maneuvers. Imagine we need to launch a space vehicle from the orbital station into the same circular orbit as that of the station, but there is to be an angular distance of $180°$ between the vehicle and the station. In other words, they are to orbit in the same circle but at opposite ends of its diameter. How can this be done?

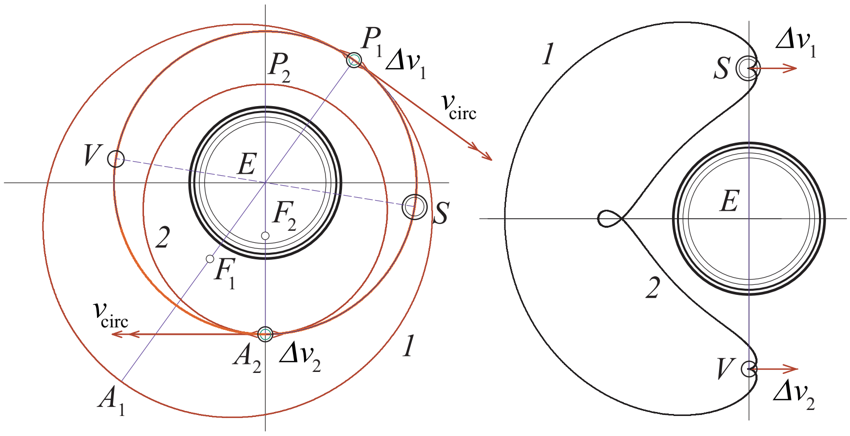

Figure 6. Transitional geocentric elliptical orbits (outer orbit *1* and inner orbit *2*) with periods $3/2\,T_0$ and $3/4\,T_0$, respectively (left), and the corresponding trajectories of relative motion in the reference frame associated with the orbital station (right).

In order to transfer the space vehicle to the opposite point of the circular orbit, an intermediate elliptical orbit with a definite period of revolution (say, $3/2\,T_0$ or $3/4\,T_0$) is required. To use the first possibility, after undocking from the station, an additional velocity Δv_1 must be imparted to the space vehicle in the direction of the orbital motion. Let this be done at some point P_1 (left-hand panel of Figure 6). This velocity Δv_1 appends to the circular velocity $v_{\rm circ}$, and the vehicle starts to move along an outer transitional elliptical orbit *1*, which grazes the initial circular orbit at perigee P_1. Its apogee lies at the opposite point A_1.

The period of revolution along this new orbit depends on its major axis P_1A_1. For our purpose, this period must equal $3/2\,T_0$, where T_0 is the period of the station. If this is the case, the station covers exactly one and a half of its circular orbit during one revolution of the space vehicle along its elliptical orbit (curve *1* in Figure 6). That is, the space vehicle reaches the common point P_1 of the two orbits (circular and elliptical) just at the moment when the station is at the diametrically opposite point of the circular orbit. At this moment the on-board rocket engine must be used for the second time in order to quench the excess velocity of the vehicle. Obviously, the additional velocity of the same magnitude Δv_1 but of the opposite direction is required. After this both the station and vehicle move in the same circular orbit being all the time at the opposite ends of its diameter (say, points S and V in Figure 6).

The right-hand panel of Figure 6 shows the vehicle's trajectory in its motion relative to the station (curve *1*). Starting to move forward from the station S in the direction of additional velocity Δv_1, the vehicle very soon turns upward and then backward. In this frame of reference, almost all its motion from S to final point V is retrograde. After quenching the excess of velocity over the circular one, the vehicle remains stationary in this frame at the antipodal point V for an indefinitely long time.

To calculate the required value of the additional velocity Δv_1 for the maneuver, we can use the conservation laws of energy and angular momentum, and Kepler's third law. Details of the calculation can be found in Appendix III. If the period T must equal $3/2\,T_0$, velocity at the perigee P_1, according to Eq. (23), must be $\sqrt{2-(2/3)^{2/3}}\,v_{\rm circ}$ = $1.1121\,v_{\rm circ}$, whence $\Delta v_1 = 0.1121\,v_{\rm circ}$.

The second above-mentioned possibility of transition to the opposite side of the circular orbit requires an inner intermediate elliptical orbit with the period $3/4\,T_0$. In the simulation shown in Figure 6 such an orbit is used for the vehicle's way back to the station. The backward additional velocity Δv_2 for the maneuver may be imparted to the vehicle at an arbitrary time moment, say, when it passes through point A_2 (see the left-hand panel of Figure 6). This point becomes the apogee of the inner transitional orbit (curve *2*). After two revolutions along this orbit the vehicle meets at this point the station, which covers during this time exactly one and a half revolution along its circular orbit. To equalize velocities of the vehicle and station, one more thrust of the same magnitude Δv_2 in the direction of orbital motion is required at point A_2. Then soft docking with the station becomes possible.

We note the extraordinary trajectory of vehicle's motion relative to the station (curve *2* on the right-hand panel of Figure 6). The vehicle traces the small loop of this trajectory while moving near the apogee A_2 of its geocentric orbit after one revolution. The required value of the vehicle's velocity at point A_2 and of additional velocity Δv_2 for this maneuver can be calculated with the help of Eq. (23) of Appendix III: the velocity must be equal to $\sqrt{2-(4/3)^{2/3}}\,v_{\rm circ}$ = $0.8880\,v_{\rm circ}$, whence $\Delta v_2 = 0.1120\,v_{\rm circ}$. We see that both transitions (through outer and inner elliptical orbits) require almost the same magnitude of the additional velocity: $\Delta v_2 \approx \Delta v_1$.

6. Concluding Remarks

Navigation in space is quite different from what we are used to due to our experience gained here on the Earth's surface. Flying a spacecraft have nothing in common with flying an aircraft. Maneuvers in space are complicated by the free-falling of orbiting bodies in the central field of the Earth's gravity. To design a space mission, we must take into account the fundamental laws of physics as they apply to orbital motions. The choice of maneuvers suitable for a specific space flight is restricted by numerous requirements.

In this paper we presented an elementary approach to selection of possible optimal orbital maneuvers for several different space flights. In particular, safe landings of spacecraft, transitions between circular orbits, rendezvous and soft docking of spacecraft with orbital stations are considered qualitatively. For the most fuel-efficient maneuvers, the additional velocity must be directed tangentially to the orbital velocity of the vehicle. This condition provides Hohmann's transfers along semielliptic transitional trajectories. The required characteristic velocities for the maneuvers are calculated with the help of conservation laws. Extraordinary trajectories of the spacecraft relative motion are illustrated by simulations. The relevant software [9] allows us also to simultaneously observe on the computer screen the spacecraft motion relative to the Earth and relative to the orbital station in a convenient time scale.

Appendix I: Trajectories of a Landing Module

For a safe return to the Earth, a landing module should approach the dense strata of the atmosphere at a very small angle with the horizon. A steep descend is dangerous because air resistance causes rapid heating of the module and, in the case of a manned spacecraft, because the astronauts may experience overloads of large g-factors. Therefore the descending trajectory should just graze the upper atmosphere.

We calculate here the additional velocity Δv for two possible impulse maneuvers to transfer the landing module from an initial circular orbit into a suitable descending trajectory: (i) the change in velocity is directed tangentially, antiparallel to the orbital velocity, and (ii) the change in velocity is directed radially, perpendicular to the orbital velocity.

An additional velocity transfers the space vehicle from the initial circular orbit to an elliptical orbit. One of the foci of the ellipse is located, in accordance with Kepler's first law, at the center of the Earth.

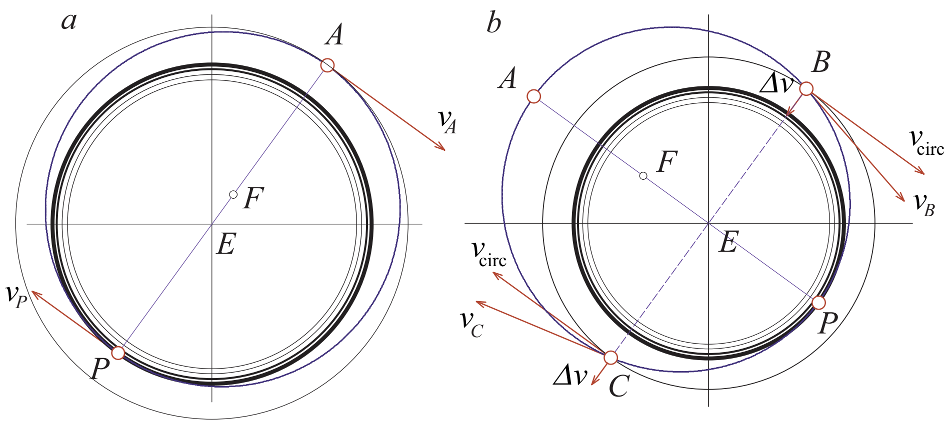

Figure 7. Possible maneuvers to transfer the landing module from a circular orbit to a trajectory grazing the planet: *a* – by additional velocity directed against the orbital velocity; *b* – by a transverse additional velocity.

In case (i), the short-term impulse thrust of the rocket engine changes only the magnitude of the orbital velocity, preserving its direction. Therefore, at the point where the rocket engine operates (point A in Figure 7, a) the descending elliptical orbit has a common tangent with the original circular orbit. This point A is the apogee of the elliptical orbit. Its perigee is located at the opposite end P of the major axis, that passes through A and the center of the Earth E. At this point P the ellipse should graze the atmosphere.

To calculate the additional velocity Δv (the characteristic velocity) that is necessary for the transition from the circular orbit to this descending elliptical trajectory, we make use of the conservation laws for energy and angular momentum.

We let $v_A = v_{\rm circ} - \Delta v$ be the velocity at the apogee A of the elliptical orbit (here $v_{\rm circ}$ is the constant velocity in the original circular orbit), and v_P be the velocity at the perigee P, where the ellipse grazes the globe (see Figure 7, a). Then we write the laws of the conservation of energy and angular momentum for these points A and P:

$$\frac{v_A^2}{2} - \frac{GM}{r_0} = \frac{v_P^2}{2} - \frac{GM}{R}; \qquad r_0 v_A = R v_P. \tag{7}$$

Here r_0 is the radius of the original circular orbit, R is the Earth's radius (including the atmosphere), and M is the mass of the Earth. Substituting v_P from the second equation into the first, we obtain:

$$v_A^2\left(1 - \frac{r_0^2}{R^2}\right) = \frac{2GM}{r_0}\left(1 - \frac{r_0}{R}\right). \tag{8}$$

Dividing both parts of Eq. (8) by $(1 - r_0/R)$, we find the required value v_A of the velocity at the apogee of the elliptical orbit:

$$v_A = \sqrt{\frac{2GM}{r_0}}\frac{1}{\sqrt{1 + r_0/R}} = v_{\rm circ}\sqrt{\frac{2}{1 + r_0/R}}. \tag{9}$$

We have expressed the first radical in Eq. (9) in terms of the circular velocity $v_{\rm circ}$ for the original orbit: $v_{\rm circ} = \sqrt{GM/r_0}$. To find the value of the required change in velocity, we subtract v_A from the circular velocity $v_{\rm circ}$. This yields for Δv the cited above expression, Eq. (1), which was used for producing the simulation shown in Figure 1).

In case (ii) the additional velocity imparted to the space vehicle is directed radially downward, transversely to the orbital velocity, and both the magnitude and direction of the velocity change. Therefore the new elliptical orbit intersects the original circular orbit at point B (see Figure 7, b) at which the additional velocity Δv is imparted to the landing module. For a soft landing, the new elliptical trajectory of descent must also graze the Earth (the upper atmosphere) at the perigee P of the ellipse.

The laws of conservation of the energy and the angular momentum for points B and P in this case can be written as follows:

$$\frac{v_{\rm circ}^2 + (\Delta v)^2}{2} - \frac{GM}{r_0} = \frac{v_P^2}{2} - \frac{GM}{R}; \qquad v_{\rm circ} r_0 = v_P R. \tag{10}$$

Here the velocity v_P at the perigee, as well as the additional velocity Δv, clearly have values different from those in Eq. (7). We note that the constant areal (sectorial) velocity in Eq. (10) for the descending elliptical trajectory has the same value as it does for the original circular orbit because an additional radial impulse from the rocket engine does not change the angular momentum of the landing module.

Substituting $v_P = v_{\rm circ} r_0/R$ into the first of Eqs. (10) and taking into account that $GM/r_0 = v_{\rm circ}^2$, we get:

$$(\Delta v)^2 = v_{\rm circ}^2 \left(\frac{r_0}{R} - 1\right)^2. \tag{11}$$

Next, substituting $r_0 = R + h$ in this equation, we finally obtain

$$\Delta v = \pm \frac{h}{R} v_{\rm circ}, \tag{12}$$

the value given by Eq. (3).

The two possible signs in Eq. (12) mean that the additional velocity to the landing module can be imparted not only downward, but also vertically upward. In both cases the landing module will be transferred to the trajectory that just grazes the Earth (see Figure $7, b$). It is clear from considerations of symmetry that in both cases the required additional velocity Δv has the same magnitude. However, to land on the Earth at the same point P, the upward impulse must be imparted to the landing module at a different point of the original circular orbit (point C in Figure $7, b$, which is opposite to point B). The angular distance between point C of the transition to the elliptical orbit and point P of the landing in this case is 270° (three quarter of a revolution). The module at first rises higher. Then, only after it passes through the apogee of its elliptical orbit (point A in Figure $7, b$), does it begin to descend towards the Earth's surface.

Appendix II: Hohmann's Transitions and Space Rendezvous

The laws of the conservation of energy and angular momentum, together with Kepler's laws of motion in a central Newtonian gravitational field, can be used in calculating the maneuvers required for a planned space flight between two circular orbits, and for an approximate calculation of an interplanetary flight.

Next we consider a semielliptic Hohmann's transition between two circular orbits. We assume for definiteness that we wish to launch a spacecraft from an orbital station that moves around a planet in a circular orbit of radius r_0 into an outer circular orbit of radius, say, $2r_0$. After the spacecraft remains for some time in this new orbit, it is to return to the station and dock to it. The simulation experiment for such maneuvers is described in Section 3 (see Figure 4). Here we present the calculations for the required characteristic velocity and for the time moments (for the backcount) at which the maneuvers must take place.

The ellipse of the semielliptic transitional trajectory that ensures the most economical transition (in expending rocket fuel) grazes both the initial circular orbit (from the outside) and the final circular orbit (from the inside). Hence the perigee distance from the center of the planet equals r_0, the radius of the initial orbit, and the apogee distance equals $2r_0$, the radius of the final circular orbit. To calculate the velocity v_0 that the spacecraft must have at perigee of the semielliptic transitional trajectory, we can use Eq. (9), replacing R in it with r_A:

$$v_0 = v_{\rm circ}\sqrt{\frac{2}{1 + r_0/r_A}}. \tag{13}$$

Here r_A is the apogee distance of the transitional elliptical orbit from the center of the planet. To find the required additional velocity Δv_1 for the first maneuver, we subtract from v_0, Eq. (13), the circular velocity v_{circ} which the spacecraft already has after undocking from the station:

$$\Delta v_1 = v_{\text{circ}} \left(\sqrt{\frac{2}{1 + r_0/r_A}} - 1 \right). \tag{14}$$

Substituting $r_A = 2r_0$, we obtain from Eq. (14) $\Delta v_1/v_{\text{circ}} = 2/\sqrt{3} - 1 = 0.1547$.

The spacecraft comes to the apogee with a velocity v_A, whose value is related to the velocity v_0 at the perigee, Eq. (13), through the law of the conservation of angular momentum (Kepler's second law):

$$v_0 r_0 = v_A r_A.$$

For $r_A = 2r_0$ we find, with the help of Eq. (13), $v_A = v_0/2 = 0.577\, v_{\text{circ}}$. To transfer the spacecraft from the elliptical orbit to the circular orbit of radius $2r_0$, we must increase the velocity at apogee by a second jet impulse. The circular velocity in a given central Newtonian gravitational field is inversely proportional to the square root of the radius of the circular orbit. For the orbit of radius $2r_0$, the circular velocity equals $v_{\text{circ}}/\sqrt{2} = 0.707\, v_{\text{circ}}$, where v_{circ} is the circular velocity for the original orbit of radius r_0. Subtracting from this value the velocity $v_A = 0.577\, v_{\text{circ}}$, at which the spacecraft reaches the apogee of the elliptical orbit, we find the additional velocity Δv_2 required for the second maneuver: $\Delta v_2/v_{\text{circ}} = 0.707 - 0.577 = 0.130$.

Next we calculate the time moments at which these maneuvers take place. We can do this with the help of Kepler's third law. The semimajor axis a of the elliptical orbit equals $(r_0 + r_A)/2 = (3/2)r_0$. We call the period of revolution along the original circular orbit (orbit of the station) T_0. Then the period for the elliptical orbit equals $(a/r_0)^{3/2}\, T_0 = 1.5^{3/2}\, T_0 = 1.837\, T_0$. If we assume $t_1 = 0$ for the first jet impulse, the second jet impulse must be imparted to the spacecraft after a lapse of one-half the period for the elliptical orbit, that is, at $t_2 = 0.9186\, T_0$.

During the lapse of time $t = t_2 - t_1$ taken for the transition, the radius vector of the station rotates through an angle $(2\pi/T_0)t$ radians. Since the radius vector of the spacecraft turns during this semielliptic transition through the angle π, at the instant of the second maneuver the spacecraft lags behind the station by an angle $\alpha = (2\pi/T_0)t - \pi = 2\pi(0.9186 - 0.5) = 2\pi \cdot 0.4186$ radians.

After the spacecraft remains for a while in its new circular orbit, it is to return to the orbital station. The optimal return path between the two circular orbits is again semielliptic. The additional velocity Δv_3 in the jet impulse that transfers the spacecraft from the outer orbit to the semielliptic transitional trajectory is directed against the orbital velocity. It is clear from symmetry that in magnitude the additional velocity this time must be exactly the same as for the preceding transition from the elliptical trajectory to the outer circular orbit, that is, $\Delta v_3 = \Delta v_2 = 0.130\, v_{\text{circ}}$. And when the spacecraft reaches the perigee of the elliptical trajectory where it grazes the inner circular orbit, one more jet impulse is necessary to quench the excess velocity. This time the additional velocity Δv_4 must have the same magnitude as it does for the first transition from the initial circular orbit to the semielliptic trajectory: $\Delta v_4 = \Delta v_1 = 0.1547\, v_{\rm circ}$.

However, the return journey of the spacecraft is complicated by the fact that it is not sufficient to simply transfer the spacecraft to the original inner circular orbit. The spacecraft must reach the grazing point of the transitional semielliptic trajectory and the inner circular orbit just at the moment when the orbital station arrives at this point. To ensure the rendezvous, we must choose a proper moment for the transition from the outer orbit to the semielliptic return path. What should the system configuration be at this moment?

During the direct transition to the outer orbit, the spacecraft lagged behind the station by an angle $\alpha = 2\pi \cdot 0.4186$ radians (α is the angle between the radius vectors of the station and the spacecraft at $t = t_2$). The journey back takes place during the same lapse of time as does the journey out. Consequently, in order to meet with the station, the spacecraft must begin its journey back at that moment when the station is behind the spacecraft by the same angle α.

Letting T be the period of revolution of the spacecraft along the outer circular orbit, it follows from Kepler's third law that $T = 2^{3/2}\, T_0 = 2.83\, T_0$, since the radius of the outer orbit is $2r_0$. Calling $\Delta\omega$ the difference between the angular velocity $2\pi/T_0$ of the station and the angular velocity $2\pi/T$ of the spacecraft, we have that $\Delta\omega = (2\pi/T_0) \cdot 0.646$. The angular distance $\beta(t)$ between the station and the spacecraft at an arbitrary time $t > t_2$ is determined by the expression:

$$\beta(t) = \Delta\omega(t - t_2) + \alpha, \tag{15}$$

since at $t = t_2$ this angular distance equals α. To calculate the time t_3 suitable for starting the return journey, we require that at the moment the station be behind the spacecraft by α. Consequently, the angle β given by Eq. (15) should be made equal to $2\pi n - \alpha$, where n is an integer:

$$2\pi n - \alpha = \Delta\omega(t_3 - t_2) + \alpha. \tag{16}$$

Since $\alpha = 2\pi \cdot 0.4186$ radians, we find from Eq. (16) that the time $t_3 - t_2$ during which we can stay in the outer circular orbit is given by:

$$t_3 - t_2 = T_0(n - 0.8372)/0.646. \tag{17}$$

For $n = 1$ Eq. (17) gives $t_3 - t_2 = 0.252\, T_0$. During this interval the spacecraft covers only a small portion of the outer orbit. And so if the spacecraft is to remain longer, we let $n = 2$ in Eq. (17) to find that $t_3 - t_2 = 1.7987\, T_0$. The period of revolution for the outer circular orbit equals $2^{3/2}\, T_0 = 2.83\, T_0$, and so with $n = 2$ the spacecraft covers a considerable part of the orbit. If we are satisfied with this duration (otherwise we can take $n = 3$ or more, say, $n = 4$), the third maneuver must be performed at $t_3 = t_2 + 1.7987\, T_0 = 2.7174\, T_0$. Adding the duration $0.9186\, T_0$ of motion along the semielliptic trajectory, we find the moment t_4 at which the rendezvous of the spacecraft with the station occurs: $t_4 = 3.636\, T_0$. At this moment the fourth jet impulse of a magnitude $\Delta v_4 = \Delta v_1 = 0.1547\, v_{\rm circ}$ must be imparted to the spacecraft in order to equalize its velocity with the orbital velocity of the station.

The above discussion illustrates how space maneuvers are calculated using Kepler's laws and the laws of conservation of energy and angular momentum. These calculations can be tested by using the simulation programs [9]. Figure 4 described in Section 3 is obtained by such a simulation. It illustrates the particular maneuvers calculated above.

Appendix III: Period of Revolution along an Elliptical Orbit

For some problems of designing a space flight, the crucial issue is the period of revolution of the spacecraft along a transitional elliptical orbit. Next we calculate the additional tangential velocity that must be imparted to the space probe after its undocking from the station in order to transfer the craft to an elliptical orbit with the required period of revolution.

We can express this period for the elliptical orbit through the length of its major axis with the help of Kepler's third law. Therefore first of all we should find the major axis for a given value of the additional velocity imparted to the spacecraft. Writing down the conservation laws of energy and angular momentum for the apogee and perigee of the elliptical orbit (like the orbit shown in Figure $7, a$, but with the perigee P not necessarily on the surface), we obtain a relationship between velocity v_A at the apogee and distance r_P towards the perigee. This relationship is just given by Eq. (8), if we replace R in it with arbitrary distance r_P to the perigee:

$$v_A^2\left(1-\frac{r_0^2}{r_P^2}\right)=\frac{2GM}{r_0}\left(1-\frac{r_0}{r_P}\right). \tag{18}$$

We can find the desired distance from the center of the planet to the perigee, r_P, by solving this quadratic equation. There is no need in reducing it to canonical form and using the standard formulas for the roots. Expressing the difference of squares in the left-hand side of the equation as the product of the corresponding sum and difference, we see at once that one of the roots is $r_P = r_0$. This root corresponds essentially to the initial point (to apogee A). This irrelevant root appears because one of the conditions used for obtaining the equation, namely that the velocity vector be orthogonal to the radius vector, is satisfied also for the initial point (as well as for the perigee).

In order to find the second root, the root that corresponds to the perigee, we divide both sides of Eq. (18) by $(1 - r_0/r_P)$ and express in it $2GM/r_0$ through the circular velocity ($GM/r_0 = v_{\rm circ}^2$) for the initial point A (see Figure $7, a$). This yields for the distance r_P to the perigee of the orbit:

$$r_P=\frac{r_0}{2(v_{\rm circ}/v_0)^2-1}. \tag{19}$$

This expression is convenient for determination of parameters of the elliptical orbit in terms of the initial distance r_0 and the initial transverse velocity v_0.

For the semimajor axis a of the elliptical orbit Eq. (19) yields the following expression:

$$a=\frac{1}{2}(r_0+r_P)=\frac{r_0}{2}\frac{1}{1-v_0^2/(2v_{\rm circ}^2)}. \tag{20}$$

If the initial velocity equals the circular velocity, that is, if $v_0 = v_{\rm circ}$, Eq. (20) gives $a = r_0$, since the ellipse becomes a circle, and the semimajor axis coincides with the radius of the orbit. If $v_0 \to \sqrt{2}v_{\rm circ}$, that is, if the initial velocity approaches the escape velocity, Eq. (20) gives $a \to \infty$: the ellipse is elongated without limit. If $v_0 \to 0$, equation (20) gives

$a \to r_0/2$: as the horizontal initial velocity becomes smaller and smaller, the elliptical orbit shrinks and degenerates into a straight segment connecting the initial point and the center of force. The foci of this degenerate, flattened ellipse are at the opposite ends of the segment.

Using Eq. (20), we can express the square of the initial velocity v_0 of the spacecraft at the common point of the two orbits in terms of the semimajor axis a:

$$v_0^2 = v_{\text{circ}}^2 \left(2 - \frac{r_0}{a}\right). \tag{21}$$

Next we can express in Eq. (21) the ratio r_0/a in terms of the desired ratio of the period T_0 of the station to the period T of the spacecraft in its elliptical orbit with the semimajor axis a. We do this with the help of Kepler's third law:

$$\frac{r_0}{a} = \left(\frac{T_0}{T}\right)^{2/3}. \tag{22}$$

Hence, to have a desired period T of revolution along a new elliptical orbit after undocking at point A (see Figure $7, a$), velocity of the spacecraft must be changed by a rocket impulse to the following value v_0:

$$v_0 = v_{\text{circ}}\sqrt{2 - (T_0/T)^{2/3}}. \tag{23}$$

Appendix IV: Approximate Differential Equations for the Relative Motion

Linearized differential equations for the relative motion of orbiting bodies are derived in [10]. The non-inertial frame of reference is used whose origin lies in the station. The z-axis of this frame points perpendicularly to the plane of the orbit; the x-axis lies in the plane of the orbit and extends radially outward, away from the center of the Earth; and the y-axis is parallel to the orbital velocity of the station, $\mathbf{v}_{\text{circ}}$. This frame rotates about z-axis with the angular velocity $\Omega = 2\pi/T_0$, where T_0 is the period of revolution of the station along its circular orbit.

The approximate equations, Eqs. (5) in [10], valid for small spatial distances between the spacecraft and the station (much smaller than the linear dimensions of the orbit), are as follows:

$$\begin{aligned} \ddot{x} &= 3\,\Omega^2 x + 2\Omega\,\dot{y}, \\ \ddot{y} &= -\,2\,\Omega\,\dot{x}, \\ \ddot{z} &= -\,\Omega^2\,z. \end{aligned} \tag{24}$$

Here x, y, and z are the coordinates that determine the position of the spacecraft relative to the station, and $\dot{x}$, $\dot{y}$, and $\dot{z}$ are the components of the relative velocity. Next we solve these equations for the situation described in Section 4: the spacecraft initially is in the same circular orbit with the station, but behind it through a small distance L. Assuming $t = 0$ at the moment of the thrust, we have $x(0) = 0$, $y(0) = -L$, $z(0) = 0$. (see Figure 8). To reach the station (which stays at the origin), the craft at $t = 0$ gets the initial velocity $\Delta v = L/(3T_0) = L\Omega/6\pi$ relative to the station, directed against the orbital velocity: $\dot{x}(0) = 0$, $\dot{y} = -\Delta v$, and $\dot{z} = 0$.

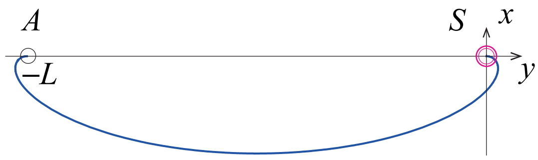

Figure 8. Trajectory of the docking spacecraft relative to the orbital station.

For these initial conditions, the particular solution to the system of the linearized equations of motion, Eqs. (24), can be written as follows:

$$x(t) = \frac{L}{3\pi}\,(\cos\Omega t - 1),$$
$$y(t) = \frac{L}{6\pi}\,(3\,\Omega t - 4\sin\Omega t) - L, \qquad (25)$$
$$z(t) = 0.$$

We can treat this solution as a periodic motion (with the period $T_0 = 2\pi/\Omega$) of the spacecraft along the ellipse

$$x(t) = \frac{L}{3\pi}\,(\cos\Omega t - 1), \quad y(t) = -\frac{2L}{3\pi}\,\sin\Omega t, \qquad (26)$$

whose semiaxes are $L/(3\pi)$ and $2L/(3\pi)$ respectively, with simultaneous uniform motion of the ellipse in y-direction with the velocity $L\Omega/2\pi = L/T_0$. During one period T_0 the ellipse displaces in y-direction (along the orbit) through distance L.

The trajectory given by Eqs. (26) for the time interval $0 < t < T_0$ is shown in Figure 8. The relative motion of the spacecraft starts at point A, located through distance L behind the orbital station S, and after the lapse of time $T_0 = 2\pi/\Omega$ ends at the origin: the craft approaches the station S. We can compare this approximate curve with the exact trajectory of relative motion shown in the right-hand panel of Figure 5, which is produced by a computer simulation.

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