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Regular Keplerian motions in classical many-body systems

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Abstract. A clear and simple physical approach to the explanation of exact particular solutions of the classical many-body problem is suggested. When the motion of individual bodies coupled by mutual gravitational forces in a many-body system occurs along conic sections, each body can be treated as moving not under the pull of the other moving bodies, but rather under a stationary central inverse-square gravitational field. These solutions describing possible amazingly simple (Keplerian) many-body motions are illustrated by computer simulations. Some pedagogical and philosophical aspects of the problem are discussed.

1. Introduction: simple and complicated motions of celestial bodies

Exact analytic solutions to the differential equations of motion are remarkable for the simplicity of motions described by these solutions. In particular, the classical Kepler problem of a body in the Newtonian inverse-square central gravitational field (and of two interacting bodies coupled by mutual gravitation) allows such analytic solutions predicting simple motions along conic sections. Unfortunately, exact solutions are seldom encountered in physics. When there are perturbing interactions (gravitational forces produced by other bodies, deviations of mass distributions from exact spherical symmetry, additional non-gravitational forces, etc), the equations of motion become non-integrable. The mathematical analysis of such perturbed motions is immensely complicated. The marvel of closed orbits that are found in Keplerian motion, as well as their wonderful simplicity, vanishes.

For example, the distortion of the Earth's gravitational field from spherical symmetry causes the actual orbit of a satellite to differ from an ellipse. The real trajectory is a complex curve, generally not closed and not lying in a plane. After a revolution, the satellite does not return to the same spatial point. The trajectory of an equatorial satellite looks like a multi-petalled flower whose leaves gradually fill out the annular region enclosed between the two concentric circles. For a hypothetical oblate planet with a somewhat exaggerated axial distortion, a possible trajectory of a satellite is shown in figure 1.

When the perturbing forces are small compared to the main gravitational force, one can use approximate analytic methods. Keplerian motion, whose distinctive feature is the striking mathematical simplicity of possible trajectories, can be assumed in this case as the zeroth approximation to the actual motion. We can consider that the small perturbations cause relatively slow variations of the parameters characterizing the corresponding Keplerian orbit, and try to obtain analytic expressions for these slow variations. Lagrange has called such an ellipse with varying parameters, grazing the actual trajectory, the *osculating orbit*. In figure 1 osculating ellipses are shown (by thin lines) for three points (S, A and B) of the trajectory.

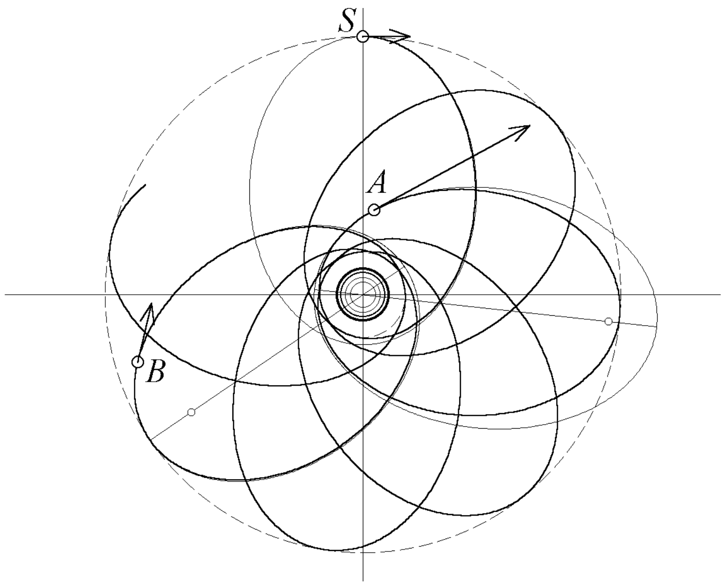

Figure 1. Precession of the orbit of an equatorial satellite orbiting an oblate planet.

If we imagine that the perturbations suddenly vanish, all parameters of the osculating ellipse remain constant during the subsequent motion, and the body traces the ellipse which touches the actual trajectory at the given point. This unperturbed Keplerian motion, for at least some part of the osculating ellipse, is very close to the actual motion. The actual (perturbed) motion in figure 1 is characterized by two different (generally incommensurate) periods: one for the radial periodic motion between maximum and minimum distances (for the variation of the radius vector magnitude), and the other for the radius vector angular rotation in a full circle. The 'wonder' of a closed elliptical orbit (generated by the pure inverse-square central force) is provided by exact coincidence of these two periods for arbitrary initial conditions.

When it is inadmissible to regard the perturbations as small ones, as, for example, in the general case of the three-body problem, it is impossible to obtain even approximate analytic solutions. In other words, there exist no general formulas that describe the motion of the bodies and that permit the calculation of their positions from arbitrary initial conditions. Even for the restricted three-body problem (in which the mass of one of the bodies is negligible compared to the masses of the other two and hence the motion of the two massive bodies is Keplerian), there is no general analytic solution for the motion of the light body. For some values of parameters of the system and/or initial conditions, the motion of the light body is irregular, seemingly random (chaotic), in spite of the deterministic character of the problem. An example of chaotic motion of a satellite orbiting a massive planet that orbits a star is illustrated in figure 2 showing the simulation of motion both in the 'heliocentric' and 'geocentric' frames of reference.

It may happen that after several revolutions about the planet the gravitational attraction of the star pulls the satellite from the planet's 'embrace,' and the satellite becomes an independent planet orbiting the star along an almost elliptical Keplerian orbit that is slightly perturbed by the planet. It may also happen that such a satellite lost by the planet, after several independent revolutions about the star, is again captured by the planet. In figure 2 such a 'restitution' occurs after approximately a 'year' (one revolution of the planet around the star) of the satellite's independent existence. Similar exchanges of the satellite with the planet and the star in this 'game of space basketball' may be repeated many times. However, these extraordinary space voyages of the satellite eventually end by its falling into the planet or star, or by its ejection from the system.

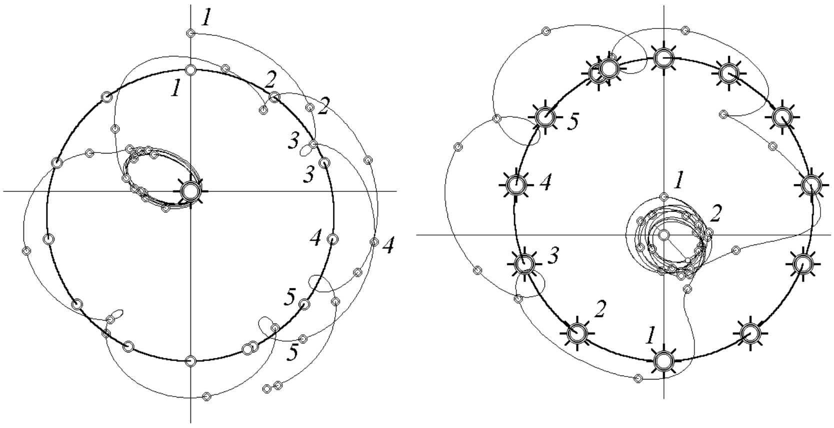

Figure 2. Trajectory of a satellite orbiting in turn a planet and star in the heliocentric (left) and in the geocentric (right) frames of reference. Identical numbers refer to the same instants of time.

Chaotic behaviour of a nonlinear system (governed by simple deterministic laws) which we observe in this example is related to the extreme sensitivity of the differential equations describing the system to the initial conditions: a very small initial difference may result in an enormous change in the future state and long-term behaviour of the system. Celestial dynamics gives one of the numerous examples of chaos in physics. We may suppose that in this case the absence of an analytical solution reflects probably the complexity of the possible motions of the system rather than the weakness of the analytic capability of the mathematics. Many examples of such complex motions can be found in the simulation programs of the package *Planets and satellites* [1] developed recently by the author. These simulations allow us to observe and study many fascinating phenomena that can occur in a system of three or more bodies attracted to one another by gravitational forces. Their motion delights the eye and challenges our intuition.

For the planets of our solar system, Kepler's laws give a good zeroth-order approximation because the masses of planets are small compared to the mass of the Sun, and the planets are separated from one another by large distances. That is, with good precision we can neglect the forces of gravitation between the planets and consider their motion to be governed only by their attraction to the Sun. Hence, because of the structure of our solar system, the motion of each of the planets is rather simple. But in a double-star system the motion of a planet can be very complicated. Many stars in our Galaxy are multiple systems—double and triple stars, unlike the Sun, which is a single star. In a multiple-star system, stable planetary orbits are also possible and it is conceivable that a community of animated, thinking creatures could arise on such a planet. Because trajectories of planets in a double-star system are very complicated, it would be an immensely difficult problem for astronomers among those creatures to establish the kinematical laws of planetary motion in the double-star system (Kepler), and even a much more difficult problem would be to discover that these complex kinematical laws are generated by the simple inverse-square law of gravitational attraction to each of the stars (Newton). Our civilization has advantage of a planet orbiting a single star. Mankind has been lucky to travel so fast along the thorny road of knowledge.

Moreover, we can even suppose without irony that Newton's law of gravitation would have been more difficult to discover if Kepler's model of planetary motion had been quickly invalidated by more accurate astronomical observations than those of Tycho Brahe. We can regard this page in the history of science as a surprising piece of evidence that a greater precision of the experimental data obtained too early can impede the scientific progress.

The three-body problem is frequently mentioned in various intermediate mechanics texts as an example of extreme complexity of possible motions generated by simple and precise physical laws. However, since Lagrange it is well known that in generally unsolvable many-body problems there exist several particular solutions describing simple Keplerian motions of the bodies. It may seem a real wonder that such an unexpectedly simple finite subset of motions falls out of the continuous set of tremendously complex general three-body motions. And these simple solutions certainly should allow an equally simple physical explanation.

There is a serious lack of relevant information regarding these simple solutions in standard texts on general physics and even on celestial mechanics. Several papers on the subject published during the last decades (see, for example, [2]) deal only with the triangular libration points in the restricted circular three-body problem. In serious advanced courses of celestial mechanics the simple result concerning the Lagrangian points is presented in an enormous heap of extremely complex formulas and therefore leaves little aesthetic satisfaction: we expect that simple and beautiful results certainly deserve simple and clear ways of their derivation.

In this paper we suggest a simple physical explanation for exact particular solutions to the many-body problem. We claim that in all cases in which the motion of individual bodies (coupled by mutual gravitational forces in a many-body system) occurs along conic sections, each body can be treated as moving not under the pull of the other moving bodies, but rather in a stationary central gravitational field whose strength is inversely proportional to the square of the distance of the body from the centre of mass of the system. Under certain conditions this effective gravitational field can be stationary in spite of the fact that it is created by moving bodies. We illustrate this idea first by the simplest example of the two-body problem.

2. The two-body problem

In most textbooks on mechanics the two-body system generally is not referred to as a many-body system because the problem of relative motion of two interacting bodies (irrespective of the physical nature of the interaction) mathematically is equivalent to the problem of motion of a virtual single body with the reduced mass $\mu = m_1m_2/(m_1 + m_2)$ under a stationary central force equal to the force of interaction between the actual bodies. The solution to this problem actually describes the motion of one body relative to the other. Under the inverse-square gravitational force of interaction, this relative motion of the bodies occurs along a conic section and obeys Kepler's laws. Knowing that if one body moves, say, in an ellipse about the other (a binary star), we can show that they both move synchronously in homothetic ellipses about the centre of mass of the whole system. As the bodies move, they are always at the ends of a rotating straight line that passes through the common focus of their orbits located at the stationary centre of mass (figure 3). The linear dimensions of these similar elliptical orbits are inversely proportional to the masses of the bodies.

Therefore, the concept of the reduced mass enables us to regard the two-body problem as one-body. This traditional approach, being quite correct and mathematically simple, may seem especially amazing to some students and cause confusion since it allows us to treat the non-inertial frame of either of the bodies as inertial. The explanation of this apparent inconsistency is all too subtle for most students who study physics at an introductory level. Moreover, the business of transforming from one frame to another in this case can be also rather confusing. (After all, both Copernicus and Galileo had difficulty getting the world to accept such ideas.)

However, dealing with the two-body problem, it is possible to use a somewhat different approach that is free of the difficulties mentioned. We consider the motion of each body in the inertial centre-of-mass frame of reference. Since the force of gravity between the bodies lies at each instant along the line joining the bodies, the force vectors are always directed through the centre of mass. In order to explain why the motions of the bodies relative to the centre of mass obey Kepler's laws and occur along conic sections, it is sufficient to show that each of the bodies coupled by mutual gravitation can be treated as moving not under the pull of the other moving body, but rather in a stationary central gravitational field whose strength diminishes

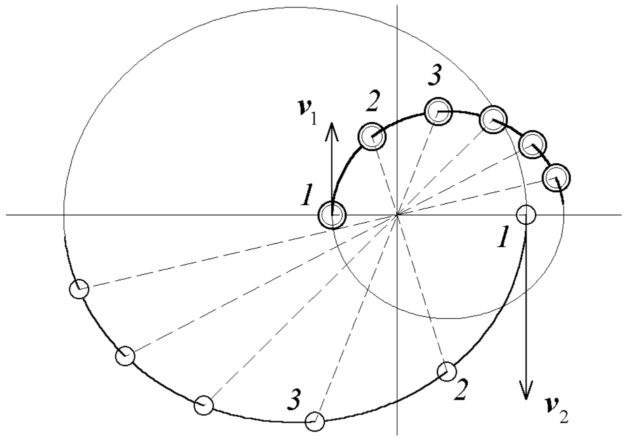

Figure 3. Trajectories traced by components of a double star in the centre-of-mass reference frame. The simultaneous positions of the bodies are marked by the same numbers.

as the square of the distance from the centre of mass. The source of the field (located at the stationary centre of mass) is characterized by some effective mass $M_{\rm eff}$.

Indeed, let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be the radius vectors denoting momentary positions relative to the centre of mass of the bodies with masses m_1 and m_2 respectively. Then $m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = 0$, and $r_1 + r_2 = [1 + m_1/m_2]r_1$. Therefore, in the formula for the gravitational force $\boldsymbol{F}_1$ exerted on the first body, we can express the distance $(r_1 + r_2)$ between the bodies in terms of the sole distance r_1 of the first body from the centre of mass:

$$\boldsymbol{F}_1 = -G\frac{m_1 m_2}{(r_1 + r_2)^2}\frac{\boldsymbol{r}_1}{r_1} = -G\frac{m_1 (M_1)_{\rm eff}}{r_1^2}\frac{\boldsymbol{r}_1}{r_1} \qquad (M_1)_{\rm eff} = \frac{m_2^3}{(m_1 + m_2)^2}. \quad (1)$$

Thus, in the inertial centre-of-mass frame, the motion of the first body in the two-body system should be exactly the same as in a central inverse-square gravitational field created by a stationary source of mass $(M_1)_{\rm eff}$ (in the absence of the second body). As we know, such a motion obeys Kepler's laws.

Similar considerations can be applied to the other body of the two-body system: the gravitational influence of the first body can be replaced by the stationary source of an effective mass $(M_2)_{\rm eff} = m_1^3/(m_1 + m_2)^2$. It remains only to prove that these Keplerian motions of both bodies occur synchronously along homothetic (closed or open) complanar orbits whose linear dimensions are inversely proportional to the masses of the bodies. This follows immediately from the equation $m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = 0$, which holds for an arbitrary moment during the motion.

In another inertial reference frame (that is, as seen from aside) the bodies of a two-body system move non-uniformly along complicated wavy or looped trajectories. This apparent complexity is generated by the superposition of their rather simple periodic Keplerian elliptical motions around the centre of mass and quite simple uniform rectilinear motion alongside the centre of mass. Figure 4 shows an example of such trajectories.

3. A collinear symmetric three-body system

We start our analysis of exact particular solutions to the many-body problem with the simplest possible example. When a pair of heavy bodies have equal masses, the restricted three-body problem has an evident exact solution, provided the third body (of negligible mass) is placed exactly halfway between the members of the pair (at the centre of mass of the system), and provided its velocity relative to the centre of mass is exactly zero. Then, since the gravitational

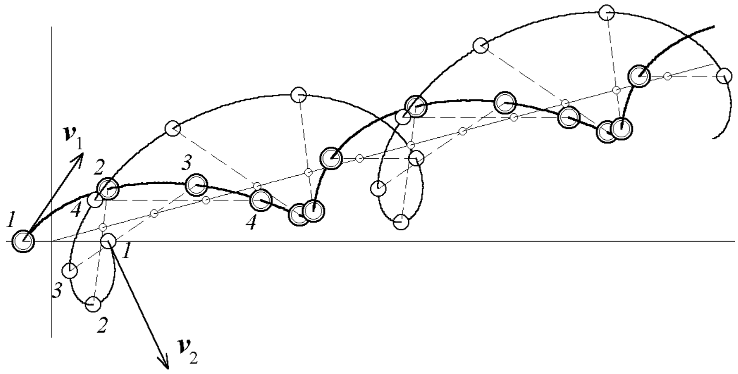

Figure 4. Trajectories of the components of a binary star as seen from aside (in an arbitrary inertial reference frame). The simultaneous positions of the bodies are marked by the same numbers.

forces exerted on the central body due to each member of the pair are equal and opposite, the central body remains at the centre of mass. This simplest particular solution is a special case of the collinear interior Lagrangian equilibrium point L_1 (see below).

This solution holds for arbitrary (not only circular) motions of the massive bodies, including cases in which they trace synchronously congruent elliptical orbits (and open parabolic or hyperbolic trajectories) with the centre of mass as their common focus. Moreover, the existence of this exact particular solution is almost evident also in the case when the central body has a finite mass. That is, the solution holds for the unrestricted three-body problem, say, for the special case of an imaginary system of two planets of equal masses that orbit a single star.

Indeed, let two massive planets (of equal masses) initially lie on the same straight line with the star at the mid-point. If the planets have equal and opposite initial velocities (in the frame of reference of the star), this configuration of the three bodies is preserved during their further motion. The only difference from the case mentioned above is that the third body of an arbitrary mass, being placed halfway between the bodies of equal masses, influences their motion because of an additional gravitational pull: the net force on either one of the pair is the sum of two gravitational forces (pointing in the same direction), one from the central body and the other from the other member of the pair. However, the net force on either body of the pair, until the symmetric configuration is violated, is in this case also inversely proportional to the square of its distance from the central body (from the centre of mass of the system).

Therefore we can consider any of the planets to move in a *stationary* Newtonian inverse-square gravitational field whose source is located at the centre of the star. The effective mass of the stationary source M_{eff} is somewhat greater than that of the star by virtue of the additional gravitational pull of the other planet. In this effective gravitational field the planet traces a closed Keplerian ellipse. The second planet moves in an equivalent effective gravitational field and traces synchronously a congruent ellipse. Although this exact solution is of no practical importance, its existence is interesting in principle and deserves discussion.

Figure 5 shows the simulation of possible simple motions in a system of two planets of equal (arbitrarily large) masses. Initially the planets A and B are on the same straight line with the star S, at equal distances on opposite sides of the star. The planets have equal and opposite initial velocities (in the heliocentric frame of reference, shown in the left-hand side of the figure). We see that in this symmetric configuration the motion of the system is regular and very simple. The star is stationary, while the massive planets trace closed orbits that are congruent ellipses with the common focus at the centre of the star. At any moment the planets are at the opposite ends of the straight line passing through the centre of the star, and their velocities are equal and opposite.

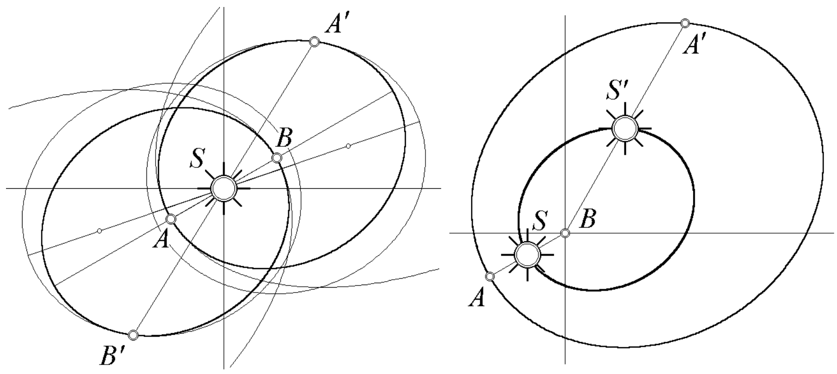

Figure 5. Simple periodic motions described by an exact particular solution to the three-body problem for a symmetric configuration of two identical planets.

The unperturbed 'heliocentric' elliptical orbits that each of the planets would trace in the absence of the other planet under the gravitational pull of the star are shown by thin lines in the left-hand side of figure 5. These osculating orbits that graze the actual elliptical orbits of the planets (thick lines) are shown for perihelia A and B (only portions of the ellipses) and for points A' and B' that are closer to aphelia (whole ellipses). The right-hand side of figure 5 shows the trajectories of the star S and planet A in the reference frame of planet B (in a somewhat smaller scale).

The net force exerted on each of the planets is formed by addition of the forces of gravitational attraction to the star and to the other planet. This resulting force is always directed toward the centre of the star, and its magnitude is inversely proportional to the square of the distance from the star:

$$F = G\frac{mM}{r^2} + G\frac{mm}{(2r)^2} = G\frac{m(M + m/4)}{r^2}. \tag{2}$$

Here G is the gravitational constant, M is the mass of the star, m is the mass of either of the planets, and r is the distance from the star to either of the planets. It follows from equation (2) that in the symmetrical configuration the motion of each of the planets occurs along a Keplerian ellipse, as if this motion were governed solely by an effective stationary central Newtonian gravitational field whose source is characterized by an effective mass of $M + m/4$.

With the help of Newton's second law, we can easily calculate the velocity of the planets for the special case of circular orbits. Equating the force given by equation (2) to the product of the planet's mass m and the centripetal acceleration v_c^2/r, we obtain for the velocity v_c of the planet in the circular orbit of a radius r:

$$v_c = \sqrt{\frac{G}{r}\left(M + \frac{m}{4}\right)}. \tag{3}$$

The period of revolution of the planets along such circular orbits is found by dividing the length of the orbit $2\pi r$ by the circular velocity v_c:

$$T = \frac{2\pi r}{v_c} = 2\pi\sqrt{\frac{r^3}{G(M + m/4)}}. \tag{4}$$

This expression is a generalization of Kepler's third law for the special case of the planetary motion under consideration. Equation (4) is equally valid for elliptic motions of the planets provided we replace the radius r with the semimajor axis a of the elliptical orbit.

The symmetric configuration of the system is preserved during the motion provided the initial velocities of the planets relative to the star are exactly equal and opposite. If the velocities differ slightly in magnitude or direction, or the distances from the star to the planets are not exactly equal, or the three bodies do not lie exactly on the same straight line, the paths of the planets sooner or later deviate from Keplerian ellipses, and these deviations progressively increase. Hence the periodic motion described by this exact particular solution of the three-body problem is unstable. Eventually the motion of the system becomes irregular and very complicated.

4. A 'round dance' of identical planets

Similar periodic exact solutions in which the bodies trace closed Keplerian orbits exist for systems of several bodies of equal masses surrounding a central body. Let n bodies ('planets') of arbitrarily large but equal masses be located at all n vertices of a regular (equilateral) polygon, and one more body (a 'star' whose mass can differ from the masses of the other bodies) be located at the centre of the polygon. In this symmetric configuration the central body is in equilibrium under the joint gravitational pull of all other bodies. The resulting gravitational force exerted on any of the other bodies (on a 'planet') by the central body and by the other planets is directed toward the centre, and its magnitude is inversely proportional to the square of the distance from the centre (or, which is the same, to the square of a linear dimension of the polygon, e.g., of the length of its side).

Therefore identical 'planets' in an equilateral configuration can trace congruent Keplerian ellipses (or even open parabolic or hyperbolic trajectories) with the common focus at the 'star,' provided the initial velocities of the planets are equal in magnitude and make equal angles with the corresponding radius vectors of the planets. The symmetric polygonal configuration of the system is preserved during the motion (figure 6).

In particular, the 'planets' can move uniformly at equal distances from one another along the same circular orbit (circumscribed about the polygon). In this case the polygon, with the planets at its vertices, rotates uniformly about its centre. For elliptical trajectories of the planets, the angular velocity of the polygon is greatest when the planets pass simultaneously through the perihelia of their orbits. In this non-uniform rotation of the polygon, the lengths of its sides vary periodically.

The upper part of figure 6 shows examples of these exact solutions for systems of three (left) and four 'planets' (right). Moving along elliptical trajectories, at any moment the bodies are at the vertices of a regular triangle and a square respectively. Thin lines show the unperturbed orbits that the 'planets' would trace in the absence of the other planets under the gravitational pull of the 'star' (about the centre of mass of the two-body system consisting of the star and the single planet). These osculating orbits are shown for the perihelia of the actual orbits and for the moments at which the 'planets' pass through the points marked by small circles.

The lower part of figure 6 shows similar systems of six and eight 'planets' of equal masses orbiting the 'star' in symmetric equilateral configurations. The regular polygon (at whose vertices the 'planets' are found) rotates non-uniformly, and the lengths of its sides vary periodically during the rotation. The osculating ellipses shown by thin lines correspond here to the unperturbed orbits of individual 'planets' in the frame of the star (rather than in the centre-of-mass frame).

5. Keplerian motions in the triangular and square equilateral configurations

We note that in the exact solutions to the many-body problem considered above, the mass of the central body can be zero. That is, a system of n bodies of equal masses located at the vertices of a regular n-sided polygon, under their mutual gravitational attraction, can perform a beautiful 'round dance' even in the absence of a central body. In particular, three bodies

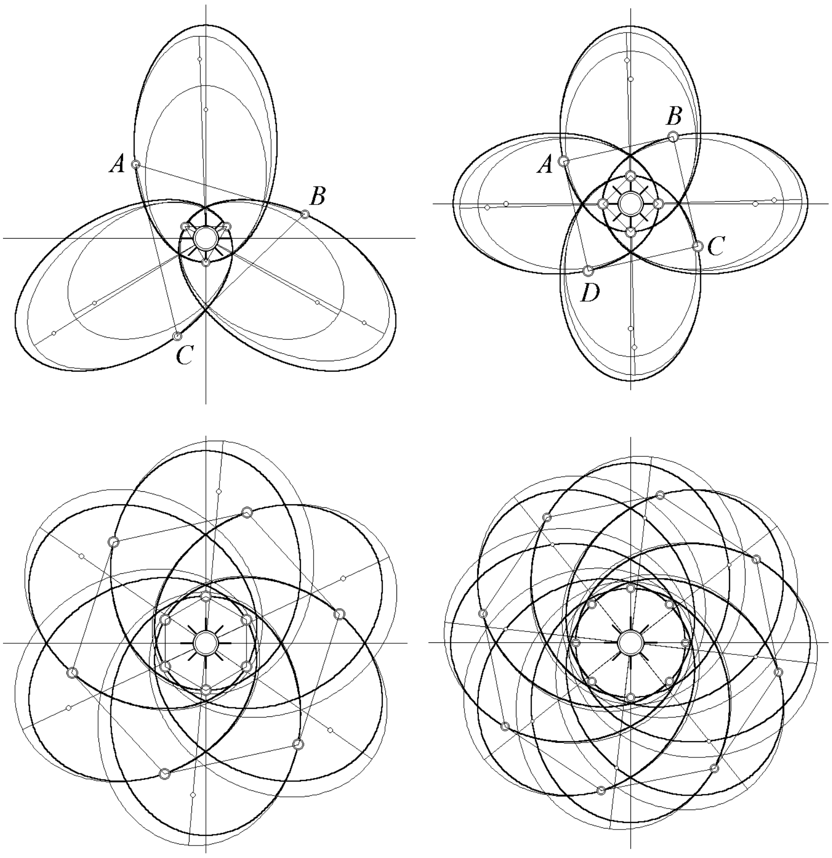

Figure 6. The polygonal systems of identical massive bodies surrounding the central body in symmetric motions described by exact particular solutions to the many-body problem.

of equal masses in the equilateral configuration can synchronously trace congruent ellipses whose major axes make angles of 120° with one another. Figure 7 shows the orbits of the three bodies A, B and S of equal masses in the centre-of-mass reference frame (left part) and in the 'heliocentric' reference frame associated with S (right part, where the scale is somewhat smaller). The thin lines grazing the actual trajectories show portions of the heliocentric orbits that each of the bodies A and B would have traced in the absence of the other (that is, only under the gravitational pull of the 'star' S) for the moment at which the planets pass through points A' and B'.

Figure 8 illustrates a regular motion in a similar equilateral configuration of four bodies of equal masses. The equilateral square configuration is preserved during the motion, but in elliptical motions the square rotates non-uniformly about the centre and 'breathes' at the rotation: the length of its sides varies periodically. Each of the bodies moves under the gravitational attraction of the other three bodies as if its motion were governed solely by a central gravitational field created by a stationary point source located at the centre of this symmetric configuration of the bodies.

The right-hand part of figure 8 shows the trajectories of three bodies in the (non-inertial) reference frame associated with one of the bodies. In this frame the square rotates about one of its vertices rather than about the centre. The lateral bodies trace smaller ellipses than the opposite body.

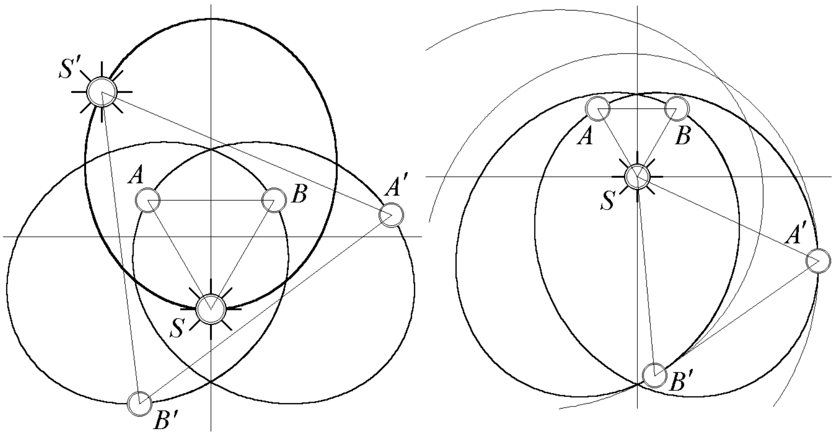

Figure 7. Regular motions of three bodies of equal masses in the equilateral configuration.

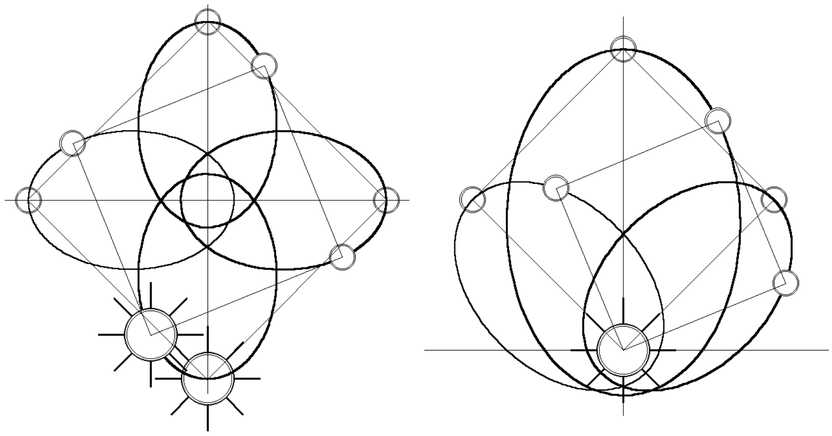

Figure 8. Regular motions of four bodies of equal masses in the equilateral configuration in the centre-of-mass reference frame (left) and in the frame of one of the bodies (right).

The equilateral configuration of three bodies is especially interesting because it can be preserved during the motion even when the masses of the bodies are different (figure 9). In the appendix we show that the total gravitational force exerted on each of the bodies by the other two bodies is directed toward the centre of mass of the system and is inversely proportional to the square of the distance from the centre of mass. We also show that the accelerations of the bodies produced by these forces are in the same ratio as are the distances of the bodies from the centre of mass. Therefore the initial equilateral configuration can be preserved during the motion, provided the initial velocities are chosen properly.

In other words, in the equilateral configuration of three bodies coupled by the gravitational forces each of the bodies can be considered as moving in an effective *stationary* central inverse-square gravitational field with the source at the centre of mass of the system, although this field is produced by the *moving* bodies. Hence the bodies can trace synchronously homothetic Keplerian ellipses with the common focus at the centre of mass of the system. Linear dimensions of these ellipses are proportional to the distances of the bodies from the centre of mass.

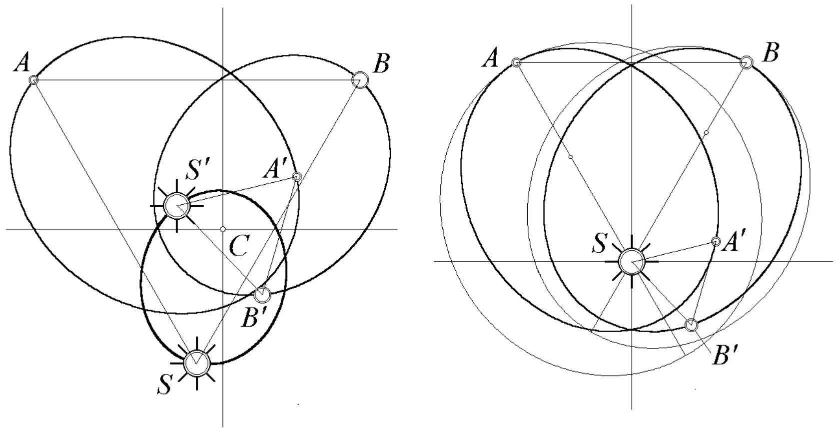

Figure 9. Regular motions of three bodies of unequal masses in the equilateral configuration.

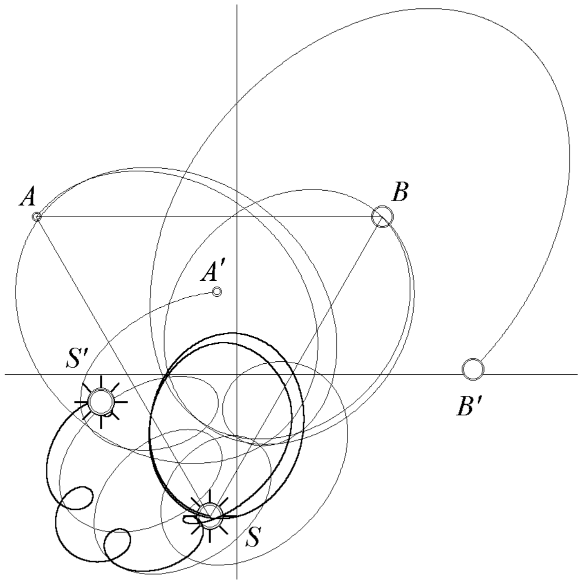

Figure 10. Transition to an irregular motion in the system whose initial motion is close to that described by the exact solution.

An example of such a simple periodic motion is shown in figure 9 ($m_A = 0.3m_S$, $m_B = 0.6m_S$). In the inertial centre-of-mass frame (left) the bodies trace homothetic elliptical orbits of different sizes and orientations. In the 'heliocentric' frame associated with the body S of greatest mass (right-hand side of the figure), the bodies A and B trace the congruent ellipses shown by thick lines. The major axes of these ellipses form an angle of 60°. The thin lines show the (non-congruent) heliocentric osculating orbits that each of the bodies A and B would have traced around S in the absence of the other body (for the moment at which A and B pass through the aphelia of their orbits).

This regular periodic motion of the three bodies is unstable with respect to (small) variations in the initial conditions that disturb the symmetry of the system. This instability of motion in the initially equilateral configuration of the bodies is illustrated by figure 10.

6. Lagrangian points in the circular restricted three-body problem

In the special case of zero mass of one of the three bodies moving in the equilateral configuration, we arrive at the triangular libration points of the restricted three-body problem, often mentioned in advanced mechanics texts and discussed in several publications [2]. It is widely known that if two massive bodies orbit each other in circles, there exist five positions at which an infinitesimal test body may be placed so that it orbits circularly about the centre of mass of the system in the same plane and at the same angular speed as do the massive bodies. That is, the whole system rotates rigidly, as if the three bodies were the points of a solid rotating uniformly about the centre of mass of the system. In other words, in the rotating reference frame associated with the line joining the primaries, the test body of a negligible mass is in equilibrium at any of these positions. These five positions are called circular libration points (or Lagrangian points) of the restricted circular three-body problem. We note that Lagrangian equilibrium points are formed by the combined gravitational forces of both massive bodies and centrifugal force of inertia.

Three of the libration points L_1, L_2 and L_3 are located on the line passing through the massive bodies (one point L_1 between the bodies). They are called collinear libration points. Each of the other two points L_4 and L_5 (triangular libration points) is located at the apex of an equilateral triangle whose base is formed by the segment joining the primaries.

The stability in the motion in the vicinity of the Lagrangian triangular libration points in the restricted planar circular three-body problem remained a subject of intense investigation in celestial mechanics for more than two centuries. It was found (see, for example, [3]) that the triangular libration points are stable for the mass ratio $\mu = m_1/(m_1 + m_2)$ (where $m_1 < m_2$) satisfying the following condition:

$$\mu(1 - \mu) < 1/27$$

with the exception of three particular values (0.0243 ..., 0.0135 ..., 0.0109 ...). That is, the triangular libration points are stable if the mass of one of the massive bodies is much smaller than that of the other (if the ratio m_1/m_2 does not exceed approximately 0.04). In the Earth–Moon system $m_1/m_2 = 0.0123$, $\mu = 0.01215$, and so its triangular libration points are stable.

In the solar system, stable triangular libration points are also formed by the combined gravitational forces of the most massive planet, Jupiter, and the Sun. There are two groups of asteroids (named Greeks and Trojans) that are trapped at Jupiter's leading and trailing triangular Lagrangian points and move around the Sun synchronously with the planet. The discussed exact solutions are of some practical interest in space dynamics because of the possibility (even if only in principle) of launching a stationary satellite located at one of the Lagrangian points in the Earth–Moon system.

The motion at collinear libration points is always unstable. One of these points lies between the massive bodies. If their masses are equal, this point is just halfway between the bodies. This case is discussed above (see figure 5). For a system with $m_B/m_A = 1/2$ (figure 11), this interior libration point is displaced from the centre of mass towards the lighter body B by 0.237 of the distance AB between the bodies. Its distance SB from the lighter body B is approximately 0.43 AB, while the distance SA from the heavier body A is 0.57 AB. In this position the resulting force of gravitational attraction by the bodies A and B is directed towards the centre of mass, and its magnitude is just sufficient to provide the satellite S with the centripetal acceleration necessary for circular motion about the centre of mass with the same angular velocity as that of the uniform rotation of the line AB joining the massive bodies. Thus, the rectilinear configuration of the system is preserved during the motion. The simultaneous positions of all the bodies are marked in figure 11 by equal numbers.

From the point of view of an observer on the heavier body A (see the right-hand side of figure 11), the lighter celestial body B, moving around A in a circular orbit, is continually eclipsed by satellite S, since the visible position of S always coincides with that of B. Similarly, an observer on B perceives the situation as a uniform revolution of S about himself in a circular

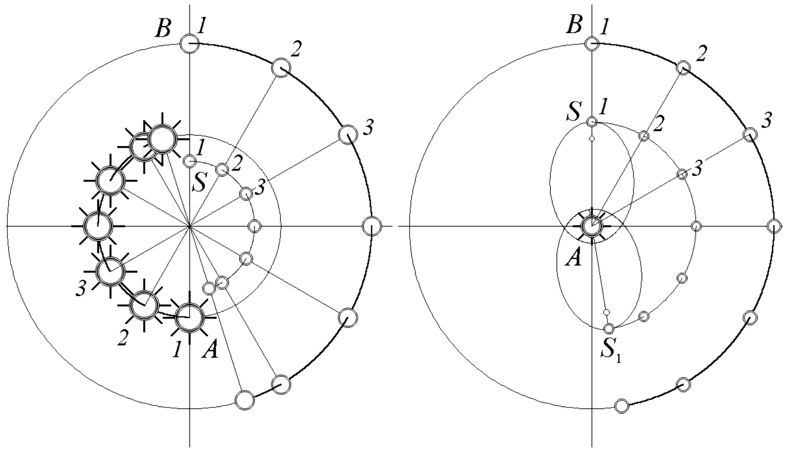

Figure 11. Motion of the satellite at the interior collinear libration point.

orbit of radius 0.43 *AB*. This revolution visually coincides with the revolution of the celestial body *A* around *B*.

The ellipses in the right-hand side of figure 11 show the osculating orbits that the satellite would trace around *A* if *B* were to suddenly vanish. (The first ellipse corresponds to the initial moment, and the second ellipse to the moment when the satellite is at the point S_1.) Indeed, the circular velocity of the unperturbed orbital motion around *A* is greater than the circular velocity of the actual motion, when the satellite is also subjected to the gravitational pull of the other body *B*. This additional pull reduces the centripetal acceleration of the satellite, and thus a smaller velocity is required for the circular motion.

Two collinear libration points lie outside the segment *AB* joining the massive bodies. For the system with equal masses ($m_B = m_A$) these points are located symmetrically at a distance of 1.198 *AB* from the centre of mass, that is, at a distance of 0.698 *AB* beyond either of the bodies. If $m_B < m_A$, one of the outer points is located closer to *B*. For $m_B/m_A = 1/2$ its distance from the centre of mass is 1.249 *AB*, so that this point of libration is separated from the lighter body *B* by a distance of 0.582 *AB*. The opposite collinear libration point is located at a distance of 1.136 *AB* from the centre of mass, so that its distance from the heavier body *A* is 0.803 *AB*.

For the Earth–Moon system, the distance of the interior libration point from the Moon is approximately 58 000 km, or 0.15 of the mean distance *AB* between the Earth and the Moon (384 400 km). The distance of the exterior point from the Moon is 65 000 km, or 0.17 *AB*. The third collinear libration point lies on the opposite (with respect to the Moon) side of the Earth. Its distance from the Earth is 380 600 km, or 0.993 *AB*.

The motion of a satellite in any of the collinear libration points (as well as the relative equilibrium in the rotating frame of reference) is unstable. An interesting example of instability of the interior libration point in the Earth–Moon system is illustrated in figure 12, which shows the motion of the system in the frame of reference associated with the Earth *E*. The initial position of the satellite *S* is very close to the libration point. If the Moon *M* were absent, the satellite (whose initial velocity is zero in the rotating frame) would have moved in the gravitational field of the Earth along an ellipse, grazing the actual circular orbit at the initial point *S*. This osculating ellipse is shown by a thin line in figure 12. The additional gravitational pull of the Moon causes the satellite to move in a circle. One more osculating ellipse is shown for the point *A* of this circular orbit.

The satellite moves with the whole system in close proximity to the libration point only during approximately one revolution (for the given initial displacement from the libration

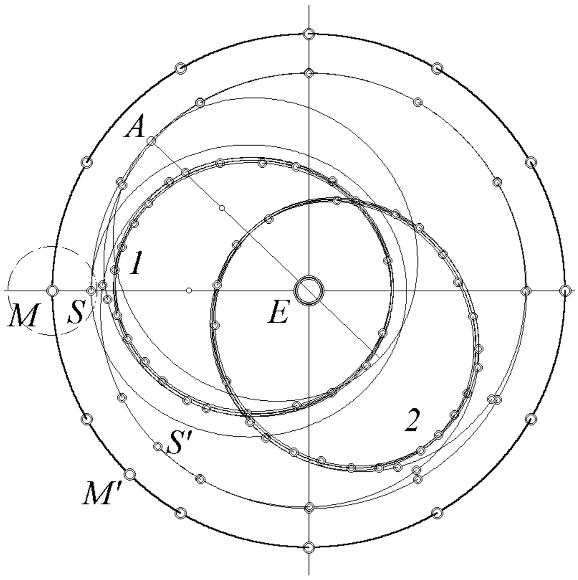

Figure 12. Motion of the Earth–Moon satellite with repeating protracted visits to the interior collinear libration point. The dashed circle around the Moon shows the sphere of gravitational action of the Moon.

point). At the end of this revolution it leaves the vicinity of the libration point and becomes an independent satellite of the Earth E. Its orbital motion around the Earth occurs in almost elliptical orbit *1* (lying inside the first osculating ellipse) which is slightly perturbed by the Moon. After several revolutions of the Moon, the satellite and the Moon occur in a close vicinity, and the gravitational field of the Moon tears the satellite away from its geocentric orbit. The satellite is again trapped at the same libration point, and for the next half-revolution of the Moon moves near this point synchronously with the whole system. Then the satellite once more escapes this unstable libration point, and again orbits the Earth independently of the Moon along another nearly elliptical orbit *2*. But after several revolutions the gravitation pull of the Moon once more returns the satellite to the vicinity of the libration point, thus making it move again during some time synchronously with the Moon. Such protracted visits of the satellite to the libration point may repeat many times. At the final moment of the simulation shown in figure 12, the Moon and the satellite are found at points M' and S' respectively.

7. Non-circular motions at the libration points

It is not widely known that exact solutions to the restricted three-body problem also exist for non-circular motions. When the two heavy bodies move around their centre of mass in elliptical orbits, the third body of negligible mass placed at one of the five libration points can also move in a closed elliptical orbit provided its initial velocity has a certain value required for this motion. During such regular non-circular motions the distances between the bodies are subjected to periodic variations while all three bodies trace homothetic ellipses with common focus at the centre of mass of the system. The program *Planets and Satellites* [1] allows us to simulate these extraordinary motions.

Figure 13 shows the periodic motion described by such an exact solution with the satellite S at the apex of the equilateral triangle ABS whose base AB is the line joining the massive bodies A and B ($m_A/m_B = 2$). The left-hand side of the figure corresponds to the inertial centre-of-mass reference frame in which all three bodies move in homothetic elliptical orbits.

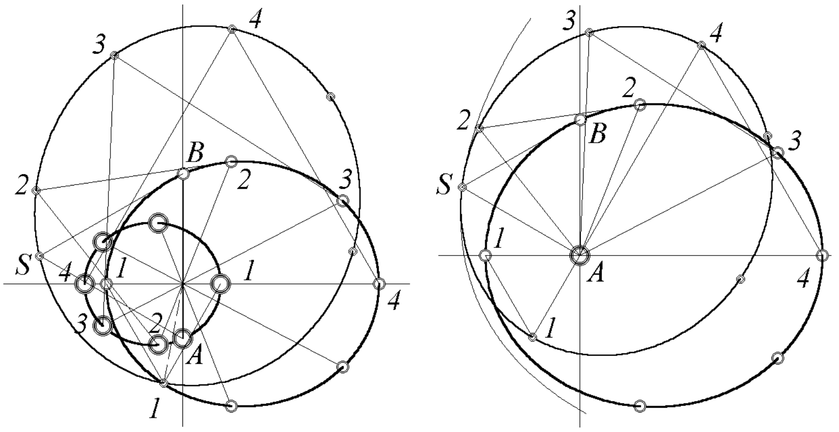

Figure 13. The periodic elliptic motions of the bodies described by an exact particular solution of the restricted three-body problem with the infinitesimal test body at the triangular libration point.

The right-hand side shows the motions of the satellite S and of body B in the frame of the heavier body A. The simultaneous positions of the bodies are marked by equal numbers.

The equilateral triangular configuration of the bodies is preserved during the motion; that is, the satellite remains at all times at the corresponding libration point. However, in contrast to the case of circular motion, here the triangle formed by the bodies rotates non-uniformly (together with line *AB* joining the bodies), and the lengths of its sides vary periodically during the motion (just as does the distance *AB* between the heavy bodies). The major axis of the ellipse traced by the satellite is at an angle with major axes of the ellipses traced by the heavy bodies. The three bodies pass simultaneously through the corresponding points of their elliptical orbits (say, through the ends of the major axes). At points marked as *1* in the figure the bodies are at their shortest distances from the centre of mass, and their angular velocity (the same for all bodies) is greatest. At the remotest points *4* the angular velocity is smallest.

In the frame of body A (the right-hand side of figure 13), body B and the satellite S move in congruent ellipses around A. The major axis of the ellipse traced by S makes an angle of 60° with the major axis of the ellipse traced by B. If body B were suddenly to vanish, the satellite would leave its elliptical orbit and move after this moment along a larger osculating ellipse. A part of this osculating ellipse grazing the actual trajectory at point S is shown in the right-hand side of figure 13.

The motion of the satellite at the interior Lagrangian point of the two heavy bodies tracing elliptical orbits is illustrated in figure 14. In this motion the satellite S remains between the primaries, on the line joining them. This line rotates non-uniformly while the bodies move along the ellipses. The position of the satellite divides the line in a constant ratio, and therefore the satellite traces an ellipse homothetic with those traced by the heavy bodies. The position of the interior Lagrangian point between the primaries depends on their masses in the same way as it does in the circular problem. For example, in a system with $m_A/m_B = 2$ (figure 14), the interior libration point is displaced from the centre of mass towards the lighter body B through 0.237 of the distance *AB* between the bodies. Its distance *SB* from the lighter body B is 0.43 *AB*, while the distance *SA* from the heavier body A is 0.57 *AB*.

From the point of view of an observer on the heavier body A (see the right-hand side of figure 14), the lighter celestial body B, moving around A in an elliptical orbit, is continually eclipsed by satellite S, since the visible position of S always coincides with that of B. The small ellipses in the right-hand side of figure 14 show the osculating orbits that the satellite would trace around A if B were to vanish suddenly. The first ellipse corresponds to the initial moment, and the second ellipse to the final moment of the simulation.

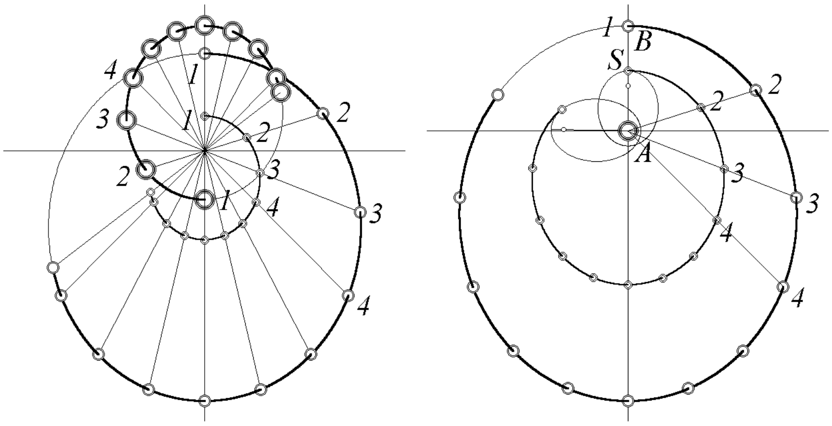

Figure 14. The periodic elliptic motions of the bodies described by an exact particular solution of the restricted three-body problem with the light body at the interior libration point.

The motion of a satellite at any of the collinear Lagrangian points is always unstable (whatever the ratio of masses of the heavy bodies may be). That is, sooner or later its simple elliptical motion inevitably transforms into irregular, chaotic orbital motion around one of the bodies, and eventually ends with an ejection of the satellite from the system or with the satellite crashing against one of the heavy bodies.

Appendix. An arbitrary three-body system in the equilateral configuration

Here we give a detailed explanation of the exact particular solutions to the unrestricted three-body problem for the triangular equilateral configuration of the bodies. We show that simple Keplerian motions are possible even when the masses of all the bodies are different.

Let the bodies *1*, *2* and *3* (of masses m_1, m_2, and m_3 respectively) be located at the vertices of the equilateral triangle with sides of length R (figure A1 shows the system with $m_1 = 0.3m_3$ and $m_2 = 0.6m_3$). We denote by $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_{13}$ the radius vectors of bodies *2* and *3* relative to body

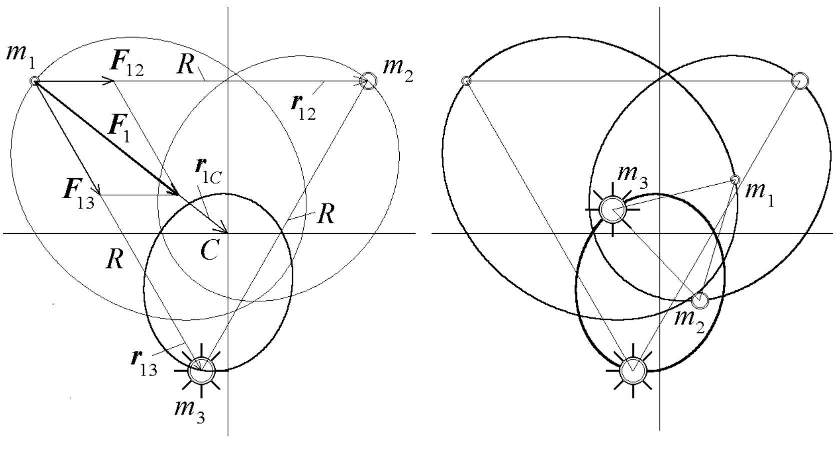

Figure A1. Regular Keplerian motions of three bodies of different masses in the equilateral configuration.

1 (that is, the vectors joining *1* with *2* and *3* respectively), and by $\boldsymbol{F}_{12}$ and $\boldsymbol{F}_{13}$ the gravitational forces exerted on *1* by *2* and *3* respectively. According to the law of gravitation,

$$\boldsymbol{F}_{12} = Gm_1m_2\frac{\boldsymbol{r}_{12}}{R^3} \qquad \boldsymbol{F}_{13} = Gm_1m_3\frac{\boldsymbol{r}_{13}}{R^3}.$$

We add $\boldsymbol{F}_{12}$ and $\boldsymbol{F}_{13}$ vectorially to find the total gravitational force $\boldsymbol{F}_1$ exerted on body *1*:

$$\boldsymbol{F}_1 = \boldsymbol{F}_{12} + \boldsymbol{F}_{13} = \frac{Gm_1}{R^3}(m_2\boldsymbol{r}_{12} + m_3\boldsymbol{r}_{13}). \tag{A1}$$

This force $\boldsymbol{F}_1$ is directed to the centre of mass C of the system. Indeed, the radius vector $\boldsymbol{r}_{1C}$ of the centre of mass relative to body *1* (the vector joining *1* with C) is given by

$$\boldsymbol{r}_{1C} = \frac{(m_2\boldsymbol{r}_{12} + m_3\boldsymbol{r}_{13})}{M} \tag{A2}$$

where $M = m_1 + m_2 + m_3$ is the total mass of the system.

With the help of equation (A2), we can express the total force $\boldsymbol{F}_1$ exerted on body *1* by the other two bodies *2* and *3* in terms of M and $\boldsymbol{r}_{1C}$:

$$\boldsymbol{F}_1 = \boldsymbol{F}_{12} + \boldsymbol{F}_{13} = \frac{GMm_1}{R^3}\boldsymbol{r}_{1C}. \tag{A3}$$

We conclude from equation (A3) that the acceleration $\boldsymbol{a}_1$ of the body *1* produced by the combined gravitation of bodies *2* and *3* is proportional to $\boldsymbol{r}_{1C}$. It is clear from symmetry that similar expressions are valid for the accelerations of the other two bodies *2* and *3* of the system:

$$\boldsymbol{a}_1 = \frac{GM}{R^3}\boldsymbol{r}_{1C} \qquad \boldsymbol{a}_2 = \frac{GM}{R^3}\boldsymbol{r}_{2C} \qquad \boldsymbol{a}_3 = \frac{GM}{R^3}\boldsymbol{r}_{3C}. \tag{A4}$$

Here $\boldsymbol{r}_{2C}$ and $\boldsymbol{r}_{3C}$ are the vectors joining the bodies *1* and *2* with the centre of mass C. Therefore the accelerations of all three bodies are directed to the centre of mass, and the magnitudes of these accelerations are proportional to the distances of the bodies from the centre of mass. This conclusion means, in particular, that the system of three bodies in the equilateral configuration can rotate as a whole (as a solid) about the centre of mass under the forces of mutual gravitation. We can find the angular velocity ω of this rotation with the help of Newton's second law. Equating the product of mass of one of the bodies (say, m_1) and the centripetal acceleration of its rotation about C to the total force exerted on this body by the other two bodies, equation (A3), we obtain

$$m_1\omega^2\boldsymbol{r}_{C1} = \frac{GMm_1}{R^3}\boldsymbol{r}_{1C} \qquad \text{whence} \qquad \omega = \sqrt{\frac{GM}{R^3}} = \sqrt{\frac{G(m_1 + m_2 + m_3)}{R^3}}. \tag{A5}$$

Such a uniform rotation of the entire system in the equilateral configuration can occur only if the initial velocities of the bodies in the centre-of-mass frame are exactly perpendicular to the radius vectors of the bodies relative the centre of mass, and magnitudes of the velocities exactly equal the product of ω and the distances of the bodies from the centre of mass. The motion is unstable. That is, if one of the above conditions is even if slightly violated, the equilateral configuration soon becomes distorted, and the motion of the bodies becomes irregular.

Uniform rotation is not the only possible regular periodic motion of the system in the equilateral configuration. We can show that the total gravitational force exerted on any of the bodies by the other two, being directed toward the centre of mass, is inversely proportional to the square of the distance to the centre of mass. Therefore under such an effectively stationary central Newtonian force (although created by the moving bodies) the body can trace a closed elliptical Keplerian orbit (or an open parabolic or hyperbolic trajectory).

In order to prove the above-mentioned property of the effective gravitational field, let us express the distance of one of the bodies (say, *1*) from the centre of mass C in terms of the distance R between any two bodies (the side of the equilateral triangle) and masses of the bodies. Calculating the square of $\boldsymbol{r}_{1C}$, equation (A2), and taking into account that the magnitudes of vectors $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_{13}$ equal R, and that the angle between them equals 60°, we find $r_{1C}^2 = R^2(m_2^2 + m_3^2 + m_2 m_3)/M^2$, whence

$$R^2 = \frac{M^2}{m_2^2 + m_3^2 + m_2 m_3} r_{1C}^2.$$

Substituting R into equation (A3), we obtain:

$$\boldsymbol{F}_1 = m_1 \frac{G(m_2^2 + m_3^2 + m_2 m_3)^{3/2}}{M^2} \frac{\boldsymbol{r}_{1C}}{r_{1C}^3}. \tag{A6}$$

Equation (A6) shows that the total gravitational force exerted on body *1* by the other two bodies is directed to the centre of mass C of the system and is inversely proportional to the square of the distance between the body and the centre of mass. Under this force the body moves in a Keplerian ellipse with one focus at the centre of mass. The same is true for the other two bodies. And since the accelerations of the bodies, according to equations (A4), are proportional to their distances from the centre of mass, all three bodies can move synchronously in homothetic ellipses with a common focus at the centre of mass, thus preserving the equilateral configuration. To simulate this regular, periodic motion, we should also choose certain initial velocities of the bodies. In the centre-of-mass reference frame, the velocities must be proportional to the distances of the bodies from the centre of mass and must make equal angles with the corresponding radius vectors.

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