Parametric Resonance in a Linear Oscillator

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Abstract

The phenomenon of parametric resonance in a linear system arising from a periodic modulation of its parameter is investigated both analytically and with the help of a computer simulation based on the educational software package PHYSICS OF OSCILLATIONS (see in the web http://www.aip.org/pas). The simulation experiments aid greatly an understanding of basic principles and peculiarities of parametric excitation and complement the analytical study of the subject in a manner that is mutually reinforcing.

The parametric excitation is studied on the example of the rotary oscillations of a mechanical torsion spring pendulum caused by periodic variations of its moment of inertia. Conditions and characteristics of parametric resonance and of parametric regeneration are discussed in detail. Ranges of frequencies within which parametric excitation is possible are determined. Stationary oscillations on the boundaries of these ranges are investigated.

1 Introduction: The Simulated Physical System

Oscillations in various physical systems may differ greatly in physical nature, but they also have much in common. It is easier to understand common laws of oscillation processes if we analyze them in the most plain and obvious examples; e.g., in mechanical systems that are accessible to direct visual observation. For this purpose, the simulation experiments described below deal with a familiar mechanical system—the torsion spring oscillator, similar to the balance device of a mechanical watch.

Left side of Figure 1 shows a schematic image of the apparatus. It consists of a rigid rod which can rotate about an axis that passes through its center. Two identical weights are balanced on the rod. An elastic spiral spring is attached to the rod. The other end of the spring is fixed. In the equilibrium position, one end of the rod points to the zero on a dial. When the rod is turned about its axis, the spring flexes. The restoring torque $-D\varphi$ of the spring is proportional to the angular displacement φ of the rotor from the equilibrium position.

The weights can be shifted simultaneously along the rod in opposite directions into other symmetrical positions so that the rotor as a whole remains balanced. However, its moment of inertia J is changed by such displacements of the weights. When the weights are shifted toward or away from the axis, the moment of inertia decreases or increases respectively. Thus the moment of inertia of the rotor is the parameter to be modulated in this simulation. As the moment of inertia J is changed, so also is the natural frequency $\omega_0 = \sqrt{D/J}$ of the torsional oscillations of the rotor. Periodic modulation of the moment of inertia can cause, under certain conditions, a progressive growth of (initially small) natural rotary oscillations of the rod suspended on the elastic spring.

The simulated physical system may seem artificial and even exotic. However, it is ideal for the study of the phenomenon of parametric resonance because it gives a very clear example of the parametric excitation in a linear mechanical system. All peculiarities of parametric resonance can be exhaustively investigated in this case, and its physical properties are

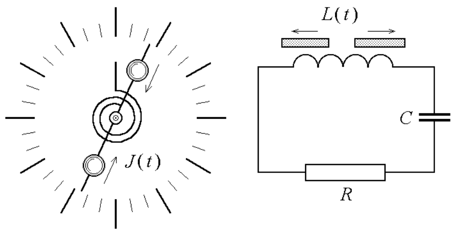

Figure 1: Schematic image of the torsion spring oscillator with a rotor whose moment of inertia is subjected to periodic variations (left), and an analogous LCR-circuit with a coil whose inductance is modulated (right).

completely explained. A mechanical system is used for the simulations primarily because its motion is easily represented on the computer screen, and it is possible to see directly what is happening. Such visualization makes the simulation experiments very convincing and easy to understand, aiding a great deal in developing our physical intuition.

Parametric excitation is also possible in an electromagnetic analogue of the spring oscillator, e.g., in a series LCR-circuit containing a capacitor, an inductor (a coil), and a resistor (right side of Figure 1). Oscillating current can be excited by periodic changes of the capacitance if we periodically move the plates closer together and farther apart, or by changes of the inductance of the coil if we periodically move an iron core in and out of the coil. Such periodic changes of the inductance are quite similar to the changes of the moment of inertia in the mechanical system considered above.

2 General Concepts

According to the conventional classification of oscillations by their method of excitation (see, e.g., [3]), oscillations are called *free* or *natural* when they occur after some initial action on an isolated system that is then left to itself. Natural oscillations are described by a homogeneous differential equation of motion: all its terms include the desired function $x(t)$ or its derivatives, and the coefficients of the equation do not depend on time. Natural oscillations in a real system gradually decay because of the energy dissipation, and the system eventually comes to rest in the equilibrium position.

Oscillations are called *forced* if an oscillator is subjected to an external periodic influence whose effect on the system can be expressed by a separate term, a given periodic function of the time, in the differential equation of motion describing the system. After a transient process is over, the forced oscillations become stationary and acquire the period of the external influence (the *steady-state* forced oscillations). When the frequency of the external force is close to the natural frequency of the oscillator, the amplitude of steady-state forced oscillations can reach significant values. This phenomenon is called *resonance*. Resonance is found everywhere in physics, and has wide and various applications.

Another way to excite non-damping oscillations consists of a periodic variation of some parameter of the system to which the motion of the system is sensitive. For example, let a restoring force $F = -kx$ arise when the system is displaced through some distance x from the equilibrium position. But in contrast to the stationary case, the parameter k changes with time because of some periodic influence: $k = k(t)$. In the differential equation of motion for such a system,

$$m\ddot{x} = -k(t)x, \qquad \text{or}$$
$$\ddot{x} + \omega^2(t)x = 0 \quad (\omega(t) = \sqrt{k/m}), \qquad (1)$$

the coefficient ω^2 of x is not constant: it explicitly depends on time. Similarly, the coefficients in the differential equation are not constant if the inertial parameter m depends on time. Oscillations in such systems are essentially different from both free oscillations, which occur when the coefficients in the homogeneous differential equation of motion are constant, and forced oscillations, which occur when an additional time-dependent forcing term is added to the right side of the equation of motion with constant coefficients.

In the case of *periodic* changes of the parameter k or ω, when $k(t+T) = k(t)$ or $\omega(t+T) = \omega(t)$, where T is the period, the corresponding differential equation, Eq. (1), is called *Hill's equation.* Oscillations in a system described by Hill's equation are called *parametrically excited* or simply *parametric.* When the amplitude of oscillation caused by the periodic modulation of some parameter increases steadily, we describe the phenomenon as *parametric resonance.* In parametric resonance, equilibrium becomes unstable and the system performs oscillations whose amplitude progressively increases.

The most familiar example of parametric resonance is given by swinging of a child on a swing. The swing can be treated as a physical pendulum whose reduced length changes periodically as the child squats at the extreme points, and straightens when the swing passes through the equilibrium position. However, the torsion spring oscillator described above is a more simple (a linear) system and hence better suits for the initial investigation of the phenomenon of parametric resonance than the pendulum with a modulated length because the latter is described by a nonlinear differential equation: The restoring torque of the force of gravity for the pendulum is proportional to the sine of the deflection angle.

The causes and characteristics of parametric resonance are considerably different from those of the resonance occurring when the oscillator responds to a periodic external force. Specifically, the resonant relationship between the frequency of modulation of the parameter and the mean natural frequency of oscillation of the system is different from the relationship between the driving frequency and the natural frequency for the usual resonance in forced oscillations. The strongest parametric oscillations are excited when the cycle of modulation is repeated twice during one period of natural oscillations in the system, i.e., when the frequency of a parametric modulation is twice the natural frequency of the system. It is evident that parametric excitation can occur only if at least weak natural oscillations already exist in the system. And if there is friction, the amplitude of modulation of the parameter must exceed a certain threshold value in order to cause parametric resonance.

Two different cases of the parametric modulation are considered in the simulation programs of the software package [1]: a square-wave variation and a smooth variation of the moment of inertia (specifically, a sinusoidal motion of the weights along the rod). In the case of the square-wave modulation, abrupt, almost instantaneous increments and decrements of the moment of inertia occur sequentially, separated by equal time intervals. We denote these intervals by $T/2$, so that T equals the period of the variation in the moment of inertia (the *period of modulation*).

A change in the moment of inertia can increase or decrease the angular velocity of the rotor. While the weights are being moved along the rod, the angular momentum of the system remains constant since no torque is needed to produce this displacement. Thus the resulting reduction in the moment of inertia is accompanied by an increment in the angular velocity, and the rotor acquires additional energy. The greater the angular velocity, the greater the increment in energy. This additional energy is supplied by the source that moves the weights along the rod. On the other hand, if the weights are instantly moved apart along the rotating rod, the angular velocity and the energy of the rotor diminish. The decrease in energy is transmitted back to the source.

For increments in energy to occur regularly and exceed the amounts of energy returned—that is, so as a whole, the modulation of the moment of inertia regularly feeds the oscillator with energy—the period and phase of modulation must satisfy certain conditions.

For example, suppose that the weights are abruptly drawn closer to each other at the instant at which the rotor passes through the equilibrium position, when its angular velocity is almost maximal. Then they are moved apart almost at the instant of extreme deflection, when the angular velocity is nearly zero. The angular velocity increases in magnitude at the moment the weights come together, and vice versa. But since the angular momentum is zero at the moment the weights move apart, this particular motion causes no change in the angular velocity or kinetic energy of the rotor. Thus the square-wave modulation of the moment of inertia with a period half the mean natural period of rotary oscillations generates a steady growth of the amplitude, provided that the phase of the modulation is chosen in the way described above.

Figure 2 shows the graphs of the angular displacement and velocity of the rotor (together with the square-wave graphs of variation of the moment of inertia) for the case in which the weights are drawn closer to and moved apart from each other twice during one mean period of the natural oscillation.

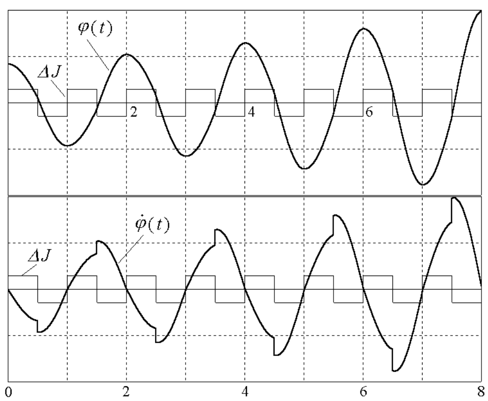

Figure 2: Graphs of the angular displacement and velocity of the rotor at square-wave modulation of its moment of inertia in the vicinity of the principal parametric resonance.

It is evident that the energy of the oscillator is increased efficiently not only when two full cycles of variation in the parameter occur during one natural period of oscillation, but also when two cycles occur during three, five or any odd number of natural periods. We shall see later that the delivery of energy, though less efficient, is also possible if two cycles of modulation occur during an even number of natural periods (resonances of even orders).

If the changes of a parameter are produced with the above mentioned periodicity but not abruptly, the influence of these changes on the oscillator is qualitatively quite similar, though the efficiency of the parametric delivery of energy (at the same amplitude of the parametric modulation) is maximal for the square-wave time dependence, because this form of modulation provides optimal conditions for the transfer of energy to the oscillating system. The case of a smooth modulation of some parameter is important for practical applications of parametric resonance. Figure 3 shows the plots of parametric oscillations of the torsion pendulum excited by a sinusoidal motion of the weights along the rod.

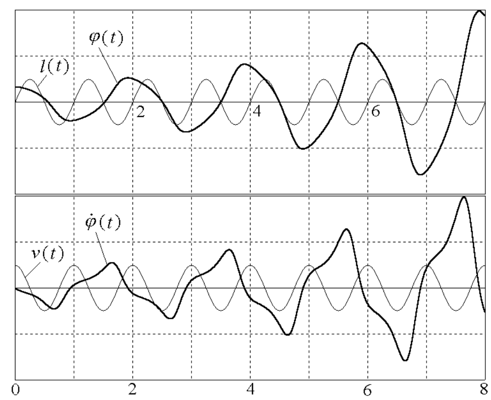

Figure 3: Graphs of the angular displacement and velocity of the rotor at a smooth modulation of its moment of inertia in conditions of the principal parametric resonance.

To provide a growth of energy during a smooth modulation of the moment of inertia, the motion of the weights toward the axis of rotation must occur while the angular velocity of the rotor is on the average greater in magnitude than it is when the weights are moved apart to the ends of the rod. Otherwise the modulation of the moment of inertia aids the damping of the natural oscillations.

Figure 4 shows the expanding phase trajectories for the parametric swinging in conditions of the principal resonance under the square-wave and smooth modulation. These phase trajectories correspond to the time-dependent graphs of increasing oscillations shown in Figures 2 and 3, respectively.

Parametric excitation is possible only if one of the energy-storing parameters, D or J (C or L in the case of LCR-circuit), is modulated. Modulation of the resistance R (or of the damping constant γ in the mechanical system) can affect only the character of the damping of oscillations. It cannot generate an increase in their amplitude.

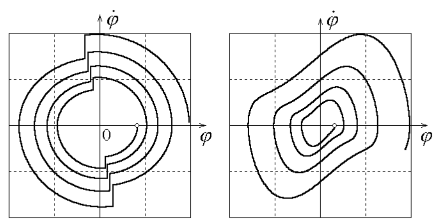

Figure 4: Phase trajectories of parametric oscillations whose time-dependent graphs are shown in the preceding figures.

3 Peculiarities of Parametric Resonance

There are several important differences that distinguish parametric resonance from the ordinary resonance caused by an external force acting directly on the system. The growth of the amplitude and hence of the energy of oscillations during parametric excitation is provided by the work of forces that periodically change the parameter. Maximal energy transfer to the oscillatory system occurs when the parameter is changed twice during one period of the excited natural oscillations. But the delivery of energy is also possible when the parameter changes once during one period, twice during three periods, and so on. That is, parametric resonance is possible when one of the following conditions for the frequency ω (or for the period T) of modulation is fulfilled:

$$\omega = 2\omega_0/n, \quad T = nT_0/2, \tag{2}$$

where $n = 1, 2, \ldots$. For a given amplitude of modulation of the parameter, the higher the order n of parametric resonance, the less (in general) the amount of energy delivered to the oscillating system during one period.

One of the most interesting characteristics of parametric resonance is the possibility of exciting increasing oscillations not only at the frequencies ω_n given in Eq. (2), but also in intervals of frequencies lying on either side of the values ω_n (in the *ranges of instability.*) These intervals become wider as the range of parametric variation is extended, that is, as the depth of modulation is increased. The dimensionless depth of modulation is defined, in the case of the rotor, as the relative difference in the maximal and minimal values of its moment of inertia: $m = (J_{\max} - J_{\min})/(J_{\max} + J_{\min})$, and in the analogues circuit, as the fractional difference in the inductance of the coil.

An important distinction between parametric excitation and forced oscillations is related to the dependence of the growth of energy on the energy already stored in the system. While for forced excitation the increment of energy during one period is proportional to the *amplitude* of oscillations, i.e., to the square root of the energy, at parametric resonance the increment of energy is proportional to the *energy* stored in the system.

Energy losses caused by friction (unavoidable in any real system) are also proportional to the energy already stored. In the case of direct forced excitation, an arbitrarily small external force gives rise to resonance. However, energy losses restrict the growth of the amplitude because these losses grow with the energy faster than does the investment of energy arising from the work done by the external force.

In the case of parametric resonance, both the investment of energy caused by the modulation of a parameter and the frictional losses are proportional to the energy stored (to the square of the amplitude), and so their ratio does not depend on the amplitude. Therefore, parametric resonance is possible only when a *threshold* is exceeded, that is, when the increment of energy during a period (caused by the parametric variation) is larger than the amount of energy dissipated during the same time. To satisfy this requirement, the range of the parametric variation (the depth of modulation) must exceed some critical value. This threshold value of the depth of modulation depends on friction. However, if the threshold is exceeded, the frictional losses of energy cannot restrict the growth of the amplitude. In a linear system the amplitude of parametrically excited oscillations must grow indefinitely.

In a nonlinear system the natural period depends on the amplitude of oscillations. If conditions for parametric resonance are fulfilled at small oscillations and the amplitude begins to grow, the conditions of resonance become violated at large amplitudes. In a real system the growth of the amplitude over the threshold is restricted by nonlinear effects.

4 The Threshold of Parametric Excitation

We can use arguments employing the conservation of energy to evaluate the modulation depth which corresponds to the threshold of parametric excitation. For the case of square-wave modulation, let us first find the increment of the rotor kinetic energy which occurs during an abrupt shift of the weights toward the axis, when the moment of inertia decreases from the value $J_1 = J_0(1+m)$ to the value $J_2 = J_0(1-m)$. During abrupt radial displacements of the weights along the rod, the angular momentum $L = J\dot{\varphi}$ of the rotor is conserved: $J_1\dot{\varphi}_1 = J_2\dot{\varphi}_2$, whence for the ratio of the angular velocities before and after the change of the moment of inertia we get $\dot{\varphi}_2/\dot{\varphi}_1 = J_1/J_2 = (1+m)/(1-m)$. It is convenient to use the expression $E_{\rm kin} = J\dot{\varphi}^2/2 = L\dot{\varphi}/2$, which gives the kinetic energy of the rotor in terms of L and $\dot{\varphi}$. For the increment ΔE of the kinetic energy we can write:

$$\Delta E = \frac{1}{2}L(\dot{\varphi}_2 - \dot{\varphi}_1) = \frac{1}{2}L\dot{\varphi}_1\left(\frac{\dot{\varphi}_2}{\dot{\varphi}_1} - 1\right) = \frac{2m}{1-m}E_{\rm kin} \approx 2mE_{\rm kin} \quad (\text{for } m \ll 1). \qquad (3)$$

If the event occurs near the equilibrium position of the rotor, when the total energy E of the pendulum is approximately its kinetic energy $E_{\rm kin}$, we see from Eq. (3) that the fractional increment of the total energy $\Delta E/E$ approximately equals twice the value of the modulation depth m: $\Delta E/E \approx 2m$.

When the frequencies and phases have those values which are favorable for the most effective delivery of energy, the abrupt displacement of the weights toward the ends of the rod occurs at the instant when the rotor attains its greatest deflection (or is very near it). At this instant the angular velocity and kinetic energy of the rotor are almost zero, and so this radial displacement of the weights into their previous positions causes no decrement of the energy.

For the principal resonance ($n = 1$) the investment of energy occurs twice during the natural period T_0 of oscillations. That is, the relative increment of energy $\Delta E/E$ during one period approximately equals $4m$. A process in which the increment of energy ΔE during a period is proportional to the energy stored E ($\Delta E \approx 4mE$) is characterized by the exponential growth of the energy in time:

$$E(t) = E_0 \exp(\alpha t). \qquad (4)$$

In this case the index of growth α is proportional to the depth of modulation m of the moment of inertia: $\alpha = 4m/T_0$. When the modulation is exactly tuned to the principal resonance ($T = T_0/2$), the decrease of energy is caused only by friction. Dissipation of energy due to viscous friction during an integral number of cycles is described by the following expression:

$$E(t) = E_0 \exp(-2\gamma t). \qquad (5)$$

where the damping constant γ equals the inverse time τ of fading of natural oscillations: $\gamma = 1/\tau$. Equation (5) yields the relative decrease of energy $\Delta E/E$ during a time interval t containing an integral number of natural periods: $\Delta E/E \approx -2\gamma t$. Equating the relative increment $4m$ of energy during one period (caused by the square-wave parameter modulation) to the relative energy losses due to friction $2\gamma T_0$, we obtain the following estimate for the threshold (minimal) value $m_{\rm min}$ of the depth of modulation corresponding to the excitation of the principal parametric resonance:

$$m_{\rm min} = \gamma T_0/2 = \pi/(2Q). \qquad (6)$$

Here we introduced the dimensionless quality factor $Q = \pi\tau/T_0 = \omega_0/(2\gamma)$ to characterize friction in the system. The parametric oscillations occurring at the threshold conditions, Eq. (6), have a constant amplitude in spite of the dissipation of energy. This mode of steady oscillations is called *parametric regeneration.* The stationary character of such oscillations is possible because on the average frictional losses of the energy are compensated for by the energy delivery from the source that makes the weights move along the rod. The mode of parametric regeneration is stable with respect to small variations in the initial conditions. However, the oscillations become fading or increasing indefinitely if we change slightly either the depth or the period of modulation or the quality factor.

For resonance of the third order (for which $T = 3T_0/2$) the threshold value of the depth of modulation is approximately three times greater than its value for the principal resonance: $m_{\min} = 3\pi/(2Q)$. In this resonance two cycles of the parametric variation occur during three full periods of natural oscillations. Radial displacements of the weights again happen at favorable moments, and so almost the same investment of energy occurs during an interval that is three times longer than the interval for the principal resonance.

When the depth of modulation exceeds the threshold value, the (averaged over the period) energy of oscillations increases exponentially in time. The growth of the energy again is described by Eq. (4). However, now the index of growth α is determined by the amount by which the energy delivered through parametric modulation exceeds the simultaneous losses of energy caused by friction: $\alpha = 4m/T_0 - 2\gamma$. The energy of oscillations is proportional to the square of the amplitude. Therefore the amplitude of parametrically excited oscillations also increases exponentially in time (see Figure 2): $a(t) = a_0 \exp(\beta t)$ with the index $\beta = \alpha/2$ (one half the index α of the growth in energy). For the principal resonance, when the investment of energy occurs twice during one natural period of oscillation, we have $\beta = 2m/T_0 - \gamma = m\omega_0/\pi - \gamma$.

In the case of the parametric growth of oscillations, energy is transmitted to the rotor by the source that makes the weights move periodically along the rod. To find the threshold of parametric excitation by a *smooth* (e.g., sinusoidal) motion of the weights along the rod, in contrast to the case of abrupt displacements, we cannot use the conservation of the angular momentum. Instead we should calculate the work done by the source during one period of oscillation and find those conditions under which this work is positive and exceeds the energy losses caused by friction.

In the adopted model of the system we let the forced motion of the weights along the rod be exactly sinusoidal, i.e., their distance l from the axis of rotation is

$$l(t) = l_0(1 + \tilde{m}\sin\omega t). \tag{7}$$

Here l_0 is the mean distance of the weights from the axis of rotation, and $\tilde{m}$ is the dimensionless amplitude of their harmonic motion along the rod ($\tilde{m} < 1$). We note that $\tilde{m}$ is the modulation depth of the distance $l(t)$, while the modulation depth m of the moment of inertia $J(t)$ is approximately twice as great ($m \approx 2\tilde{m}$ if $m \ll 1$), because the moment of inertia is proportional to the square of the distance of the weights from the axis of rotation.

The calculation of the threshold of parametric resonance for the case of the sinusoidal motion of the weights is somewhat more complicated than for the square-wave modulation considered above. Details of the calculation can be found in [1], pp. 133–135.

For $\tilde{m} \ll 1$, the moment of inertia is modulated nearly harmonically with the depth $m \approx 2\tilde{m}$. For this case we can find an approximate value for the depth of modulation of the moment of inertia at the threshold: $m = 2/Q$. This value is somewhat greater than $m = \pi/(2Q)$ given by Eq. (6) for the case of square-wave modulation, in agreement with the already mentioned qualitative conclusion that the square-wave modulation provides more favorable conditions for the transfer of energy to the oscillator from the source that moves the weights along the rod.

5 The Frequency Intervals of Parametric Excitation

The threshold for the parametric excitation of the torsion pendulum is determined above for the resonant situations in which two cycles of the parametric modulation occur during one natural period of oscillation (or during three natural periods for resonance of the third order). The estimate obtained, Eq. (6), is valid for small values of m.

For large values of the modulation depth m, the natural period needs a more precise definition. Let $T_0 = 2\pi/\omega_0 = 2\pi\sqrt{J_0/D}$ be the period of oscillation of the rotor when the weights are fixed at some middle positions which correspond to a mean value of the moment of inertia $J_0 = \frac{1}{2}(J_{\max} + J_{\min})$.

The period is somewhat longer when the weights are moved further apart: It has the value $T_1 = T_0\sqrt{1+m} \approx T_0(1+m/2)$. The period is shorter when the weights are moved closer to one another: $T_2 = T_0\sqrt{1-m} \approx T_0(1-m/2)$.

It is convenient to define the average period $T_{\rm av}$ not as the arithmetic mean $\frac{1}{2}(T_1+T_2)$, but rather as the period that corresponds to the arithmetic mean frequency $\omega_{\rm av} = \frac{1}{2}(\omega_1+\omega_2)$, where $\omega_1 = 2\pi/T_1$ and $\omega_2 = 2\pi/T_2$. So we define $T_{\rm av}$ by the relation:

$$T_{\rm av} = \frac{2\pi}{\omega_{\rm av}} = \frac{2T_1T_2}{(T_1+T_2)}. \quad (8)$$

The period T of the parametric modulation which is exactly tuned to any of the parametric resonances is determined not only by the order n of this resonance, but also by the depth of modulation m. In order to satisfy the resonant conditions, the increment in the phase of natural oscillations during one cycle of modulation must be equal to π, 2π, 3π, …, $n\pi$, … During the first half-cycle the phase increases by $\omega_1 T/2$, during the second half-cycle—by $\omega_2 T/2$, and instead of the approximate condition of resonance, Eq. (2), we obtain:

$$\frac{\omega_1+\omega_2}{2}T = n\pi, \quad T = n\frac{\pi}{\omega_{\rm av}} = n\frac{T_{\rm av}}{2}. \quad (9)$$

Thus, for a parametric resonance of some definite order n, the condition for exact tuning can be expressed in terms of the harmonic mean period $T_{\rm av}$ of the two natural periods, T_1 and T_2, defined by Eq. (8). This simple condition is $T_n = nT_{\rm av}/2$.

For moderate values of m it is possible to use approximate expressions for the average frequency and the corresponding period:

$$\omega_{\rm av} \approx \omega_0(1+\frac{3}{8}m^2), \quad T_{\rm av} \approx T_0(1-\frac{3}{8}m^2).$$

The difference between $T_{\rm av}$ and T_0 reveals itself in terms proportional to the square of the depth of modulation m.

An infinite growth of the amplitude during parametric excitation in this linear system is possible not only at exact tuning to one of resonances but also in certain *intervals* of T-values. These intervals, or the *ranges of instability,* surround the resonant values $T = T_{\rm av}/2$, $T = T_{\rm av}$, $T = 3T_{\rm av}/2$, … Generally, the width of the intervals increases with the depth m of the parameter modulation. Outside the intervals the equilibrium position of a torsion pendulum is stable, and the amplitude of oscillations does not grow.

In order to determine the boundaries of the frequency ranges of parametric instability, we can consider *stationary oscillations* of a constant amplitude that occur when the period of modulation T corresponds to one of the boundaries.

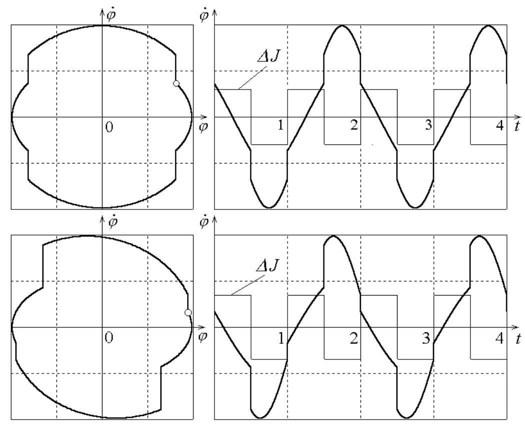

Figure 5: Stationary parametric oscillations at the lower boundary of the principal interval of instability (near $T = T_{\rm av}/2$).

For the square-wave modulation these stationary oscillations can be represented as an alternation of free oscillations with the periods T_1 and T_2, occurring during the intervals of constancy of the moment of inertia. The graphs of such oscillations are formed by joined segments of sine curves symmetrically truncated on both sides in the absence of friction, and by segments of damped sine curves of natural oscillations otherwise (see Figure 5, whose upper part corresponds to an idealized frictionless system). Periods T_1 and T_2 of these sine functions on adjacent intervals differ in accordance with the instantaneous values of the moment of inertia. For the periodic oscillations occurring at the lower boundary of the principal resonance the period T of parametric variation is a little shorter than the resonant value $T_{\rm av}/2$, i.e., a little less than a quarter of the mean natural period $T_{\rm av}$ elapses between consecutive abrupt increases and decreases of the moment of inertia. Similar phase diagrams and time-dependent graphs of stationary oscillations occurring at the upper boundary of this interval are shown in Figure 6. Here a little more than quarter of $T_{\rm av}$ elapses between the changes of the moment of inertia.

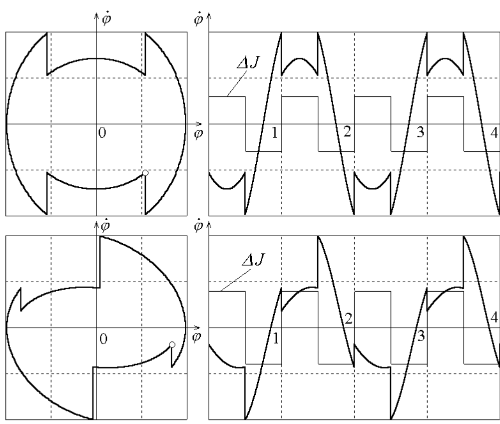

Figure 6: Stationary parametric oscillations at the upper boundary of the principal interval of instability (near $T = T_{\rm av}/2$).

The closed phase diagrams and graphs of the angular velocity $\dot\varphi(t)$ for such periodic stationary processes have the characteristic patterns shown in Figures 5 and 6. In the absence of friction the abrupt increments of the velocity occurring twice during a period are equal in a stationary process to the decrements caused by the modulation of the moment of inertia, because otherwise the oscillations will either damp or grow. With friction, stationary oscillations occur only if the increments are greater than decrements in order to compensate for the energy losses caused by friction.

To find conditions at which such stationary oscillations take place, we can write the expressions for $\varphi(t)$ and $\dot\varphi(t)$ for the adjacent time intervals during which the oscillator executes natural oscillations, and then fit these expressions to one another at the boundaries. Such fitting must provide a periodic stationary process. It is convenient to represent the motion on one interval during which the moment of inertia is $J_1 = J_0(1+m)$ as a superposition of sine and cosine (damped) waves of the frequency $\omega_1 = \omega_0/\sqrt{1-m}$ whose amplitudes are A_1 and B_1, and similarly for the adjacent interval during which $J = J_2 = J_0(1-m)$:

$$\begin{aligned} \varphi_1(t) &= (A_1 \sin\omega_1 t + B_1 \cos\omega_1 t)e^{-\gamma t}, \\ \varphi_2(t) &= (A_2 \sin\omega_2 t + B_2 \sin\omega_2 t)e^{-\gamma t}. \end{aligned} \qquad (10)$$

To determine the values of constants A_1, B_1 and A_2, B_2, we can use the conditions that must be satisfied when the segments of the $\varphi(t)$ graph are joined together, namely, the continuity of $\varphi(t)$ and the jump of $\dot\varphi(t)$ determined by the ratio $J_2/J_1 = (1-m)/(1+m)$. The two other conditions we get from the requirement of the periodicity of the stationary process. The homogeneous system of equations for A_1, B_1, A_2, B_2 which we thus obtain has a nontrivial (non-zero) solution only if its determinant is zero. This condition for the existence of a non-zero solution yields an equation for the unknown variable T, the period of modulation, which appears in coefficients of the system of equations for A_1, B_1, A_2, B_2 through the time t at which the graphs (10) are joined. Solving this equation for T, we can find the desired lower and upper boundaries, T_- and T_+, for the interval of parametric resonance as the roots of the equation.

The equation for T is derived in [1] (pp. 110–114) for an idealized frictionless system, and an approximate equation valid for the system with relatively weak viscous friction is obtained in [2] (pp. 73–76). The equation is transcendental (the variable T enters it as the arguments of sine and cosine functions) and cannot be solved analytically. In the simulation program [1] the equation for T is solved numerically by iteration, and the boundaries of the intervals of parametric resonance for arbitrary values of the depth of modulation m and the quality factor Q are displayed in the panel "Properties."

The intervals of instability that surround the first five parametric resonances are shown in the diagram (Figure 7) as functions of the depth m of the square-wave modulation. The central line of each "tongue" of the diagram shows the period $T = nT_{\rm av}/2$ that

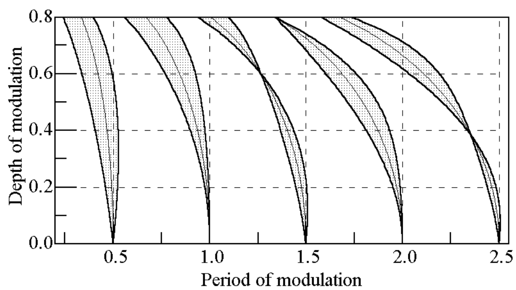

Figure 7: Intervals of parametric excitation at the square-wave modulation of the moment of inertia in the absence of friction.

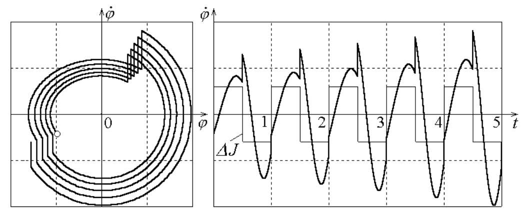

Figure 8: The phase trajectory and the graph of the angular velocity of oscillations corresponding to parametric resonance of the second order ($T = T_{\rm av}$).

corresponds to exact tuning to n-order resonance.

We note that for small values of m the intervals surrounding resonances of even orders ($n = 2$, 4) are very narrow compared to odd resonances ($n = 1$, 3, 5). To understand why resonances of even orders are so weak and narrow, we should take into account that the abrupt changes of the moment of inertia for, say, $n = 2$ resonance induce both an increase and a decrease of the angular velocity only once during each natural period. The oscillations grow only if the increment of the velocity at the instant when the weights are drawn closer is greater than the decrement occurring when the weights are drawn apart. This is possible only if the weights are shifted toward the axis when the angular velocity of the rotor is greater in magnitude than it is when they are shifted apart. Such conditions are easily fulfilled for odd resonances because the weights are shifted apart at extreme points where the velocity is zero. For $T \approx T_{\rm av}$, the mentioned conditions for the growth of oscillations can fulfill only because there is a (small) difference between the natural periods T_1 and T_2 of the rotor, where T_1 is the period with the weights shifted apart and T_2 is the period with them shifted together. This difference is proportional to the depth of modulation m and vanishes when m tends to zero. The growth of oscillations at parametric resonance of the second order is shown in Figure 8.

For small values of the depth of modulation m, it is possible to obtain approximate analytical expressions for the boundaries of the intervals of parametric resonance. For the principal resonance ($n = 1$) in the absence of friction:

$$T_{\mp} = \frac{1}{2}\left(1 \mp \frac{m}{\pi}\right) T_{\rm av}, \qquad (11)$$

(see [1], p. 115). We see from Eq. (11) that the width of the main interval ΔT is proportional to the first power of m: $\Delta T = T_+ - T_- = m/\pi$. In the presence of viscous friction:

$$T_{\mp} = \frac{1}{2}\left(1 \mp \frac{1}{\pi}\sqrt{m^2 - m_{\rm min}^2}\right) T_{\rm av}, \qquad (12)$$

where $m_{\rm min} = \pi/(2Q)$ is the threshold (minimal) value of the depth of modulation (see [2], p. 77). For the threshold conditions $m = m_{\rm min}$, and both boundaries of the interval merge. That is, the interval disappears.

Similar approximate expressions are valid for $n = 3$ resonance: In Eqs. (11)–(12) we should replace $\frac{1}{2}$ with $\frac{3}{2}$, and for the threshold depth of modulation $m_{\rm min}$ use the value $m_{\rm min} = 3\pi/(2Q)$ corresponding to $n = 3$.

For resonance of the second order ($n = 2$) in the absence of friction the boundaries are:

$$T_{\mp} = (1 \mp m^2/4)\, T_{\rm av} \qquad (13)$$

(see [1], p. 119). In the presence of viscous friction the interval shrinks:

$$T_{\mp} = \left(1 \mp \frac{1}{4}\sqrt{m^4 - m_{\rm min}^4}\right) T_{\rm av}, \qquad (14)$$

where $m_{\min} = \sqrt{2/Q}$ is the threshold value of the depth of modulation for this resonance (see [2], p. 85). In this case, the investment of energy during a period is proportional to the *square* of the depth of modulation m, while in the cases of resonances with $n = 1$ and $n = 3$ the investment of energy is proportional to the first power of m. Therefore, for the same value of the damping constant γ (the same quality factor Q), a greater depth of modulation m is required here to exceed the threshold of parametric excitation.

The interval of instability in the vicinity of resonance with $n = 2$ is considerably narrower compared to the corresponding intervals of resonances with $n = 1$ and $n = 3$. According to Eq. (13), the width $\Delta T = \frac{1}{2}m^2 T_{\text{av}}$ of this resonance in the absence of friction is also proportional only to the square of m (for $m \ll 1$), in contrast to odd resonances, for which ΔT is proportional to the first power of m.

The diagram in Figure 7 shows that with the growth of the depth of modulation m the even intervals expand and become comparable with the intervals of odd orders. We see that for some certain values of m both boundaries of intervals with $n > 2$ coincide (we may consider that they *intersect*). Thus at these values of m the corresponding intervals of parametric resonance disappear. These values of m correspond to the natural periods T_1 and T_2 of oscillation (associated with the weights far apart and close to each other), whose ratio is 2:1, 3:1, and 3:2. For the corresponding values of the modulation depth m and the period of modulation T, oscillations are steady for arbitrary initial conditions.

When there is friction in the system, the intervals of the period of modulation become narrower, and for strong enough friction (below the threshold) the intervals disappear. The diagram in Figure 9 shows the boundaries of the first three intervals of parametric resonance in the absence of friction, for $Q = 20$, and for $Q = 10$. Note the “island” of parametric resonance for $n = 3$ and $Q = 20$. This resonance disappears when the depth of modulation exceeds 45% and reappears when m exceeds approximately 66%.

For any given value m of the depth of modulation, only several first intervals (if any) of parametric resonance can exist, for which m exceeds the threshold.

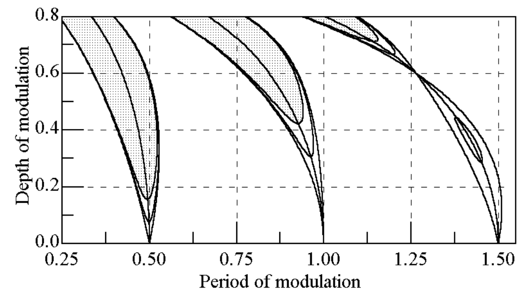

Figure 9: Intervals of parametric excitation at square-wave modulation of the moment of inertia in the absence of friction, and at weak friction (for $Q = 20$ and $Q = 10$).

6 A Smooth Modulation of the Moment of Inertia

In the investigation of smooth modulation we should rely on the differential equation rather than on the method of joining solutions for adjacent intervals that we used for square-wave modulation.

For simplicity we consider the rod itself to be very light, so that the moment of inertia J of the rotor is due principally to the weights: $J = 2Ml^2(t)$. The angular momentum $J\dot{\varphi}(t)$ changes in time according to the equation:

$$\frac{d}{dt}(J\dot{\varphi}) = -D\varphi, \qquad (15)$$

where $-D\varphi$ is the restoring torque of the spring. Substituting into Eq. (15) $l(t)$ from Eq. (7) and taking into account the expression $\omega_0^2 = D/J_0$ (where $J_0 = 2Ml_0^2$ is the moment of inertia of the rod with the weights in their mean positions), we obtain finally:

$$\frac{d}{dt}\left[(1 + \tilde{m}\sin\omega t)^2\dot{\varphi}\right] = -\omega_0^2\varphi - 2\gamma\dot{\varphi}. \qquad (16)$$

We have added the drag torque of viscous friction to the right-hand side of Eq. (16). This equation is solved numerically in the computer program [1] during the simulation of oscillations at sinusoidal motion of the weights.

In order to obtain an approximate solution that is valid up to terms of the first order in the small parameter $\tilde{m}$, we can, instead of the exact differential equation of motion, Eq. (16), solve the following approximate equation:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2(1 - 2\tilde{m}\sin\omega t)\varphi = 0. \qquad (17)$$

Equation (17) is a special case of Hill's equation, Eq. (1), with sinusoidal time dependence of the parameter. It is called *Mathieu's equation.* The theory of Mathieu's equation has been fully developed, and all significant properties of its solutions are well known. A complete mathematical analysis of its solutions is rather complicated and is usually restricted to the determination of the frequency intervals in which the state of rest in the equilibrium position becomes unstable: at arbitrarily small deviations the amplitude of incipient small oscillations begins to increase progressively in time. The boundaries of these intervals of instability depend on the depth of modulation $2\tilde{m}$. It is worth mentioning that even inside the intervals (when conditions for parametric resonance are satisfied) if φ and $\dot{\varphi}$ are exactly zero simultaneously, they remain zero. This property contrasts with the usual case of resonance in which the system is acted upon by an external force: In forced oscillations the amplitude begins to grow even from the state of rest in the equilibrium position.

We note that the application of the theory of Mathieu's equation to the simulated system is restricted to the linear order in $\tilde{m}$. For finite values of the depth of modulation $\tilde{m}$, the resonant frequencies and the boundaries of the intervals of instability for the simulated system differ from those predicted by Mathieu's equation. An approximate analytical solution to the exact differential equation of motion, Eq. (16), valid up to the terms of the second order in $\tilde{m}$ for the main resonance and resonance of the second order, is obtained in [1] (and [2]) by the method described in [4]. In particular, for the main resonance this solution gives the threshold condition which coincides with the cited above (see p. 7) condition $m_{\min} = 2/Q$ (where $m \approx 2\tilde{m}$), obtained from considerations based on the conservation of energy. For the second resonance this approximate solution gives the following threshold condition:

$$\tilde{m}_{\min} = 2/\sqrt{Q}, \qquad Q_{\min} = 4/\tilde{m}^2. \qquad (18)$$

We see that the threshold for the second parametric resonance in the case of the smooth modulation is also somewhat greater than in the case of square-wave modulation (see p. 11): compare the expression for $m_{\min}$ given by Eq. (18) with $m_{\min} = \sqrt{2/Q}$ (where $m \approx 2\tilde{m}$).

The simulation program in [1] allows us to study parametric oscillations and obtain graphs of the variables for arbitrarily large values of the depth of modulation $\tilde{m}$.

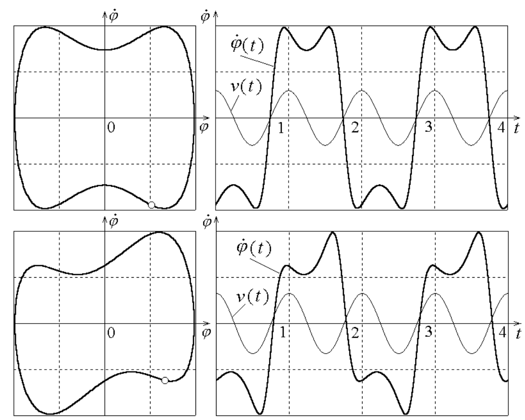

Figure 10: Stationary parametric oscillations at the upper boundary of the principal interval in the case of sinusoidal modulation.

An example of steady oscillations occurring at the upper boundary of the principal instability interval (the frequency of modulation $\omega \approx 2\omega_0$) is shown in Figure 10. Its upper part corresponds to an idealized frictionless system. We note the deviation of the shape of the plots from a sine curve caused by the contribution of higher harmonics (mainly of the third harmonic with the frequency $3\omega/2$).

It is interesting to compare these graphs of stationary parametric oscillations in the case of smooth modulation of the moment of inertia (Figure 10) with the corresponding graphs in the case of square-wave modulation (Figure 6).

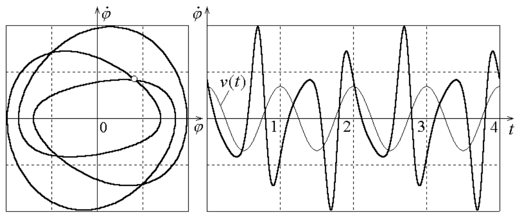

Figure 11: The phase trajectory and the plots of stationary oscillations at the threshold of the third parametric resonance

In the case of a smooth modulation of the moment of inertia, parametric resonance of the third order is weaker and narrower than that of the second order (in contrast to the case of square-wave modulation, for which at $m \ll 1$ the third-order resonance is stronger and wider than the second-order resonance). This third-order interval disappears in the presence of very small friction. Stationary oscillations at the threshold of parametric resonance of the third order are shown in Figure 11.

Summary

We discussed here a theoretical approach to the phenomenon of parametric resonance in a linear mechanical oscillator supported by its computerized experimental investigation on simple mathematical models of real physical systems. This investigation is based on simulations included in the software package PHYSICS OF OSCILLATIONS [1]. The package offers many interesting pre-defined examples that illustrate by computer simulations various properties and peculiarities of parametric resonance, thus allowing the student to appreciate the beauty of oscillatory phenomena.

The programs of the package are rather simple and at the same time flexible and sophisticated enough in order to use them, say, in advanced research projects of the more inquisitive students for exploration of new properties. Visualization of motion simultaneously with plotting the graphs of different variables and phase trajectories makes the simulation experiments very convincing and comprehensible. These simulations help greatly in developing our physical intuition and provide a good background for the study of more complicated nonlinear parametric systems like a pendulum whose length is forced to periodically change, or a rigid pendulum whose suspension point is driven periodically in the vertical direction [5].

Acknowledgments

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