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PARAMETRIC RESONANCE

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COMPUTER SIMULATION EXPERIMENTS ARE ESPECIALLY GOOD TOOLS FOR HELPING STUDENTS UNDERSTAND BASIC PRINCIPLES OF PHYSICS. IN THIS AREA, I HAVE DEVELOPED A PACKAGE OF SIMULATION PROGRAMS CALLED *PHYSICS OF OSCILLAtions* (*www.aip.org/pas*). One of its programs is suited for examining the phenomenon of *parametric resonance* in a linear system. The program simulates the parametric excitation of the rotary oscillations of a mechanical torsion-spring pendulum whose moment of inertia is subject to periodic variations.

In this installment of the "CSE in Education" department, I discuss the conditions and characteristics of parametric resonance, including parametric regeneration. Instructors and their students can also use the Physics of Oscillations package to explore other problems, such as ranges of frequencies within which parametric excitation is possible and stationary oscillations on the boundaries of these ranges. The simulation experiments complement the analytical study of the subject in a manner that is mutually reinforcing.

The simulated physical system

Oscillations in various physical systems can differ greatly in physical nature, but they also have much in common. It is easier to understand common laws of oscillation processes if we analyze them in the most plain and obvious examples, such as in mechanical systems that are accessible to direct visual observation. For this purpose, the simulation experiments this article describes deal with a familiar mechanical system—the torsion-spring oscillator, similar to a mechanical watch's balance device.

Figure 1a shows a schematic image of the apparatus. It consists of a rod on which two identical weights are balanced. The rod, to which an elastic spiral spring is attached, can rotate about an axis that passes through its center. The spring's other end is fixed. When the rod turns about its axis, the spring flexes. The spring's restoring torque $-D\varphi$ is proportional to the rotor's angular displacement φ from the equilibrium position.

The weights can shift simultaneously along the rod in opposite directions into other symmetrical positions to keep the rotor balanced as a whole. When the weights shift toward or away from the axis, the moment of inertia decreases or increases respectively. Under certain conditions, periodic modulation of the moment of inertia can cause the rod's (initially small) natural rotary oscillations to grow.

Parametric excitation is also possible in an electromagnetic analog of the spring oscillator—that is, in a series LCR circuit containing an inductor (a coil of inductance L), a capacitor C, and a resistor R (see Figure 1b). Oscillating current can be excited by periodic changes of the capacitance if we periodically move the plates closer together and farther apart, or by changes of the coil's inductance if we periodically move an iron core in and out of the coil. Such periodic changes of the inductance closely resemble the changes of the moment of inertia in the mechanical system I discussed earlier. We use a mechanical system for the simulations primarily because its motion is easily represented on the computer screen and shows directly what is happening. Such visualization makes the simulation experiments very convincing and easy to understand, greatly helping our students develop their physical intuition.

General concepts

According to the conventional classification of oscillations by their method of excitation,[1] oscillations are called *free* or *natural* when they occur after some initial action on an isolated system that is then left to itself. Natural oscillations in a real system gradually decay because of the energy dissipation, and the system eventually comes to rest in the equilibrium position.

Oscillations are *forced* if an oscillator is subjected to an external periodic influence whose effect on the system can be expressed by a separate term—a given periodic function of the time—in the differential equation of motion describing the system. After a transient process finishes, the forced oscillations become stationary and acquire the period of the external influence (*steady-state* forced oscillations).

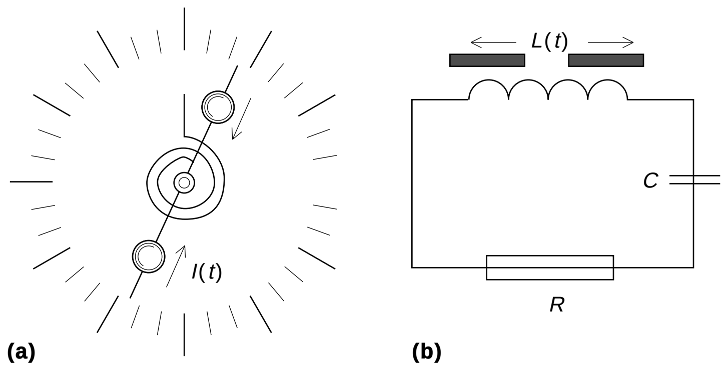

Figure 1. Computer simulation: (a) the torsion-spring oscillator with a rotor whose moment of inertia is subjected to periodic variations; (b) an analogous LCR circuit with a coil whose inductance is modulated.

When the external force's frequency is close to the oscillator's natural frequency, the amplitude of steady-state forced oscillations can reach significant values. This phenomenon is called *resonance*. Found everywhere in physics, resonance has wide and various applications.

Another way to excite nondamping oscillations consists of a periodic variation of some system parameter to which the system's motion is sensitive. For example, let a restoring force $F = -kx$ arise when the system is displaced through some distance x from the equilibrium position, and let the parameter k change with time because of some periodic influence: $k = k(t)$. In the differential equation of motion for such a system,

$$m\ddot{x} = -k(t)x,\ \ \ddot{x} + \omega^2 x = 0 \quad \left(\omega^2 = \frac{k}{m}\right), \qquad (1)$$

where the coefficient ω^2 of x is not constant: it explicitly depends on time. Similarly, the coefficients in the differential equation are not constant if the inertial parameter m depends on time. Oscillations in such systems are essentially different from both free oscillations (which occur when the coefficients in the homogeneous differential equation of motion are constant) and forced oscillations.

With *periodic* changes of parameters k or ω, the corresponding differential equation (Equation 1) is called *Hill's equation*. When the amplitude of oscillation caused by the periodic modulation of some parameter increases steadily, we refer to the phenomenon as *parametric resonance*. In parametric resonance, equilibrium becomes unstable and the system leaves it (after an arbitrarily small initial disturbance), executing oscillations whose amplitude increases progressively.

The most familiar example of parametric resonance is a child swinging on a swing. The swing is a physical pendulum whose reduced length changes periodically as the child squats at the extreme points and straightens when the swing passes through the equilibrium position. However, the torsion-spring oscillator I described earlier is a simpler (a linear) system and hence better suits for the initial investigation of parametric resonance than the pendulum with a modulated length. That's because we use a nonlinear differential equation to describe the latter: the restoring torque of the force of gravity for the pendulum is proportional to the sine of the deflection angle.

The causes and characteristics of parametric resonance are considerably different from those of the resonance occurring when the oscillator responds to a periodic external force. Specifically, the resonant relationship between the parameter's frequency of modulation and the system's mean natural frequency of oscillation is different from the relationship between the driving frequency and the natural frequency for the usual resonance in forced oscillations. The strongest parametric oscillations are excited when the cycle of modulation repeats twice during one period of natural oscillations in the system—that is, when a parametric modulation's frequency is twice the system's natural frequency. Parametric excitation can occur only if at least weak natural oscillations already exist in the system. And if there is friction, the parameter's amplitude of modulation must exceed a certain threshold value to cause parametric resonance.

The simulation programs in my software package consider two different cases of parametric modulation:[2] a square-wave variation and a smooth variation of the moment of inertia (specifically, a sinusoidal motion of the weights along the rod). With the square-wave modulation, abrupt, almost instantaneous increments and decrements of the moment of inertia occur sequentially, separated by equal time intervals. We denote these intervals by $T/2$, so that T equals the period of the variation in the moment of inertia (the *period of modulation*).

The forced motion of the weights along the rod doesn't change the angular momentum because no torque is needed to produce this displacement. Thus the resulting reduction in the moment of inertia is accompanied by an increment in the angular velocity, and the rotor acquires additional energy. The greater the angular velocity, the greater the increment in energy. This additional energy comes from the source that moves the

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Peculiarities of parametric resonance

The growth of the amplitude and hence of the energy of oscillations during parametric excitation is provided by the work of forces that periodically change the parameter. Maximal energy transfer to the oscillatory system occurs when the parameter changes twice during one period of the excited natural oscillations. But the delivery of energy is also possible when the parameter changes once during one period, twice during three periods, and so on. That is, parametric resonance is possible when one of the following conditions for the frequency ω (or for the period T) of modulation is fulfilled:

$$\omega = 2\omega_0/n, \quad T = nT_0/2, \tag{A}$$

where $n = 1, 2, \ldots$. For a given amplitude of modulation of the parameter, the higher the order n of parametric resonance, the less (in general) the amount of energy delivered to the oscillating system during one period.

Parametric resonance is possible not only at the frequencies ω_n given in Equation A, but also in intervals of frequencies lying on either side of the values ω_n (in the *ranges of instability*). These intervals become wider as the range of parametric variation is extended—as the depth of modulation increases. We define the dimensionless depth of modulation, in the case of the rotor, as the relative difference in the maximal and minimal values of its moment of inertia, $m = (I_{max} - I_{min})/(I_{max} + I_{min})$, and in the analog circuit, as the fractional difference in the inductance of the coil.

An important distinction between parametric excitation and forced oscillations relates to how energy growth depends on the energy already stored in the system. While for forced excitation, the increment of energy during one period is proportional to the amplitude of oscillations—that is, to the square root of the energy—at parametric resonance, the increment of energy is proportional to the energy stored in the system.

Energy losses caused by friction (unavoidable in any real system) are also proportional to the energy already stored. With direct forced excitation, an arbitrarily small external force gives rise to resonance. However, energy losses restrict the amplitude's growth because these losses grow with the energy faster than does the investment of energy arising from the work done by the external force.

With parametric resonance, both the investment of energy caused by a parameter's modulation and the frictional losses are proportional to the energy stored (to the square of the amplitude), so their ratio does not depend on amplitude. Therefore, parametric resonance is possible only when a threshold is exceeded, that is—when the increment of energy during a period (caused by the parametric variation) exceeds the amount of energy dissipated during the same time. To satisfy this requirement, the depth of modulation must exceed some critical value. This threshold value of the depth of modulation depends on friction. However, if the threshold is exceeded, the frictional losses of energy cannot restrict the amplitude's growth. In a linear system, the amplitude of parametrically excited oscillations must grow indefinitely.

In a nonlinear system (for example, a pendulum whose length is modulated), the natural period depends on the amplitude of oscillations. If conditions for parametric resonance are fulfilled at small oscillations and the amplitude begins to grow, the conditions of resonance become violated at large amplitudes. In a real system, nonlinear effects restrict the growth of the amplitude over the threshold.

weights along the rod. But, if the weights are instantly moved apart along the rotating rod, the rotor's angular velocity and energy diminish. The decrease in energy transmits back to the source. For increments in energy to occur regularly and exceed the amounts of energy returned—that is, so that as a whole, the modulation of the moment of inertia regularly feeds the oscillator with energy—the period and phase of modulation must satisfy certain conditions.

For example, suppose that we abruptly draw the weights closer to each other at the instant at which the rotor passes through the equilibrium position, when its angular velocity is almost maximal. Then, we move them apart almost at the instant of extreme deflection, when the angular velocity is nearly zero. The angular velocity increases in magnitude at the moment the weights come together, and vice versa. Because the angular momentum is zero at the moment we move the weights apart, this particular motion causes no change in the rotor's angular velocity or kinetic energy. Thus the square-wave modulation of the moment of inertia with a period half the mean natural period of rotary oscillations generates a steady growth of the amplitude, provided that we choose the modulation's phase as I've described.

Figure 2a shows the graphs of the rotor's angular displacement (top) and angular velocity (bottom) (together with the square-wave graphs of variation of the moment of inertia) for the case in which we draw the weights closer to and farther apart from each other twice during one mean period of the natural oscillation.

Clearly, the oscillator's energy increases efficiently not only when two full cycles of variation in the parameter occur during one natural period of oscillation, but also when two cycles occur during three, five, or any odd number of natural periods. We shall see later that the delivery of energy, although less efficient, is also possible if two cycles of modulation occur during an even number of natural periods (resonances of even orders).

If the changes of a parameter are produced with the just-mentioned periodicity but not abruptly, the influence of these changes on the oscillator is qualitatively quite similar, although the efficiency of the parametric delivery of energy (at the same amplitude of modulation) is maximal for the square-wave time dependence.

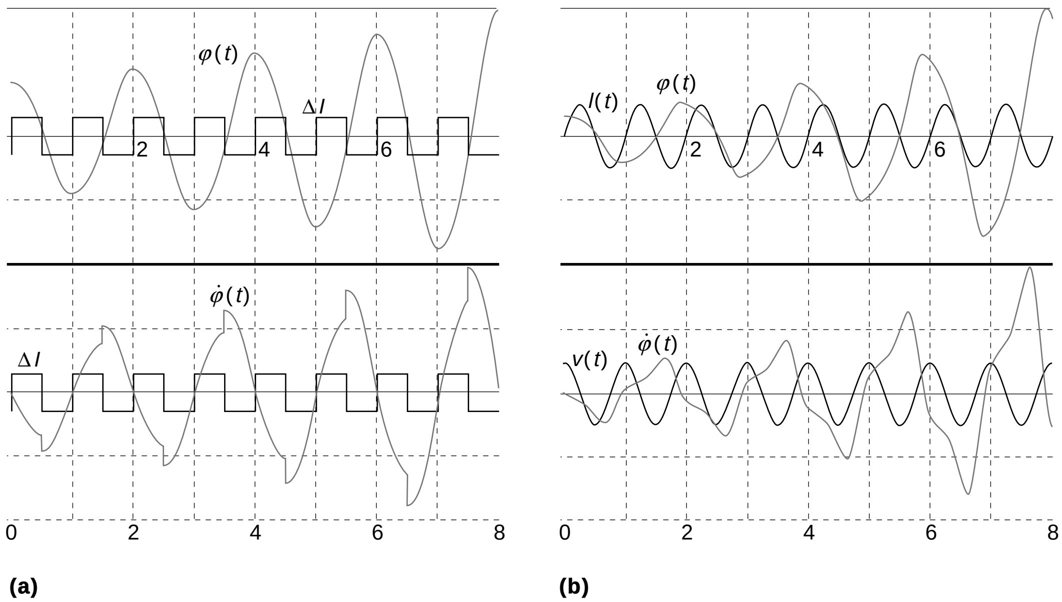

Figure 2. Graphs of the rotor's angular displacement (top) and velocity (bottom): (a) at square-wave modulation of its moment of inertia in the vicinity of the principal parametric resonance; (b) at a smooth modulation of its moment of inertia in conditions of the principal parametric resonance.

The case of a smooth modulation of some parameter is important for practical applications of parametric resonance. Figure 2b plots the parametric oscillations of the torsion pendulum excited by a sinusoidal motion of the weights along the rod.

To provide a growth of energy during a smooth modulation of the moment of inertia, the weights' motion toward the axis of rotation must occur while the rotor's angular velocity is on the average larger than it is when we move the weights apart to the ends of the rod. Otherwise the modulation of the moment of inertia aids the damping of the natural oscillations.

Figure 3 shows the expanding phase trajectories for the parametric swinging conditions of the principal resonance under the square-wave and smooth modulation. These phase trajectories correspond to the time-dependent graphs of increasing oscillations shown in Figure 2.

Parametric excitation is possible only if one of the energy-storing parameters, D or I (C or L in the case of an LCR circuit), is modulated. Modulation of the resistance R (or of the damping constant γ in the mechanical system) can affect only the character of the oscillator damping. It cannot generate an increase in the amplitude of oscillations.

The "Peculiarities of parametric resonance" sidebar discusses several important differences that distinguish parametric resonance from the ordinary resonance caused by an external force acting directly on the system.

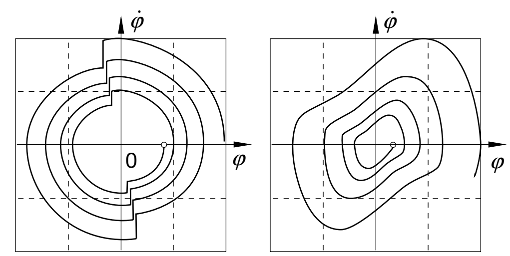

Figure 3. Phase trajectories of parametric oscillations whose time-dependent graphs are shown in Figure 2.

The threshold of parametric excitation

We can use arguments employing the conservation of energy to evaluate the modulation depth corresponding to the threshold of parametric excitation. For square-wave modulation, let us first find the increment of the rotor kinetic energy occurring during an abrupt shift of the weights toward the axis, when the moment of inertia decreases from the value $I_1 = I_0(1 + m)$ to the value $I_2 = I_0(1 - m)$. During abrupt radial displacements of the weights along the rod, the angular momentum $L = I\dot{\varphi}$ of the rotor is conserved: $I_1\dot{\varphi}_1 = I_2\dot{\varphi}_2$, whence for the ratio of the angular velocities before and after the change of the moment of inertia, we get $\dot{\varphi}_2/\dot{\varphi}_1 = I_1/I_2 = (1 + m)/(1 - m)$. For the increment ΔE of the kinetic energy, $E_{\text{kin}} = I\dot{\varphi}^2/2 = L^2/(2I)$ we can write

$$\Delta E = \frac{L^2}{2I_0}\left(\frac{1}{1-m} - \frac{1}{1+m}\right) \approx 2mE_{\text{kin}}, \qquad (2)$$

If the event occurs near the rotor's equilibrium position, when the total energy E of the pendulum is approximately its kinetic energy E_{kin}, we see from Equation 2 that the fractional increment of the total energy $\Delta E/E$ approximately equals twice the value of the modulation depth m: $\Delta E/E \approx 2m$.

When the frequencies and phases have values that are favorable for the most effective delivery of energy, the abrupt displacement of the weights toward the ends of the rod occurs at the instant when the rotor attains its greatest deflection (or is very near it). At this instant, the rotor's angular velocity and kinetic energy are almost zero, so this radial displacement of the weights into their previous positions causes no decrement of the energy.

For the principal resonance ($n = 1$), the investment of energy occurs twice during the natural period T_0 of oscillations. That is, the relative increment of energy $\Delta E/E$

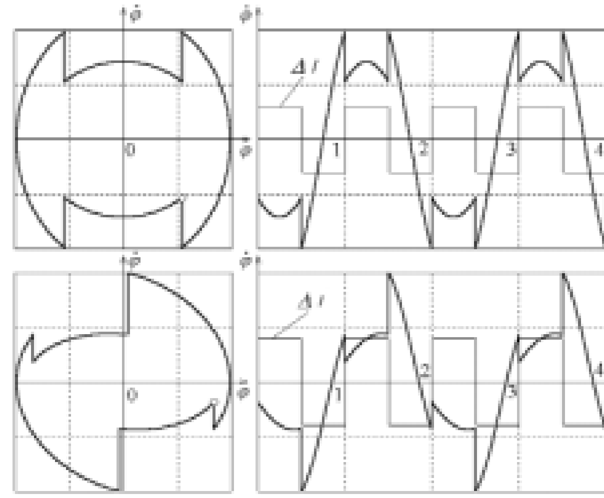

Figure 4. Stationary parametric oscillations at the upper boundary of the principal interval of instability (near $T = T_{av}/2$): (a) without friction; (b) with friction.

during one period approximately equals $4m$. A process in which the increment of energy ΔE during a period is proportional to the energy stored E ($\Delta E \approx 4mE$) is characterized by the exponential growth of the energy in time:

$$E(t) = E_0 \exp(\alpha t). \quad (3)$$

In this case, the index of growth α is proportional to the moment of inertia's depth of modulation m: $\alpha = 4m/T_0$. When the modulation is exactly tuned to the principal resonance ($T = T_0/2$), friction alone causes the decrease of energy. The following expression describes energy dissipation caused by viscous friction during an integral number of cycles:

$$E(t) = E_0 \exp(-2\gamma t), \quad (4)$$

where the damping constant γ equals the inverse time τ of fading of natural oscillations in the system: $\gamma = 1/\tau$. Equation 4 yields the relative decrease of energy $\Delta E/E$ during a time interval t containing an integral number of natural periods: $\Delta E/E \approx -2\gamma t$. We equate the relative increment $4m$ of energy during one period (caused by the square-wave parameter modulation) to the relative energy losses due to friction $2\gamma T_0$. Thus, we obtain the following estimate for the depth of modulation's threshold (minimal) value m_{min} corresponding to the excitation of the principal parametric resonance:

$$m_{min} = \gamma T_0/2 = \pi/(2Q). \quad (5)$$

We introduced the dimensionless quality factor $Q = \pi\tau/T_0 = \omega_0/(2\gamma)$ to characterize friction in the system. The parametric oscillations occurring at the threshold conditions (Equation 5) have a constant amplitude in spite of the energy dissipation. We call this mode of steady oscillations *parametric regeneration.* The mode of parametric regeneration is stable with respect to small variations in the initial conditions. However, the oscillations fade or increase indefinitely if we change slightly either the modulation's depth or period or the quality factor.

For third-order resonance (for which $T = 3T_0/2$), the depth of modulation's threshold value is approximately three times greater than its value for the principal resonance: $m_{min} = 3\pi/(2Q)$. In this resonance, two modulation cycles occur during three full periods of natural oscillations, so almost the same investment of energy occurs during an interval that is three times longer than the interval for the principal resonance.

Resonances of even orders are weaker. For example, the threshold value of the depth of modulation for the second resonance ($T \approx T_0$) equals $m_{min} = \sqrt{2/Q}$ (see the user's manual,[2] p. 85).

When the depth of modulation exceeds the threshold value, the (averaged over the period) energy of oscillations increases exponentially in time. Equation 3 again describes the energy's growth. However, the growth index α is determined by the amount by which the energy delivered through parametric modulation exceeds the simultaneous losses of energy caused by friction: $\alpha = 4m/T_0 - 2\gamma$. The energy of oscillations is proportional to the square of the amplitude. Therefore, the amplitude of parametrically excited oscillations also increases exponentially in time (see Figure 2a): $a(t) = a_0\exp(\beta t)$ with the index $\beta = \alpha/2$ (one half the index α of the growth in energy). For the principal resonance, we have $\beta = 2m/T_0 - \gamma = m\omega_0/\pi - \gamma$.

To find the threshold of parametric excitation by a *smooth* (sinusoidal) motion of the weights along the rod, we should calculate the work done (during one period of oscillation) by the source that makes the weights move periodically and determine those conditions under which this work is positive and exceeds the energy losses caused by friction. We assume that the distance l from the axis of rotation is

$$l(t) = l_0 (1 + \bar{m} \sin \omega t). \quad (6)$$

Here l_0 is the mean distance of the weights from the axis of rotation, and $\bar{m}$ is the dimensionless amplitude of their harmonic motion along the rod ($\bar{m} < 1$). Note that $\bar{m}$ is the modulation depth of the distance $l(t)$, while the modulation depth m of the moment of inertia $I(t)$ is approximately twice as great ($m \approx 2\bar{m}$ if $m << 1$), because the moment of inertia is proportional to the square of the distance of the weights from the axis of rotation.

Calculating the threshold of parametric resonance for the sinusoidal motion of the weights is somewhat more complicated than for the square-wave modulation considered above. You can find details of the calculation in the software package's user manual (pp. 133–135).[2] The calculation yields $m = 2/Q$ for the depth of modulation of the moment of inertia at the threshold condition. This value is some-

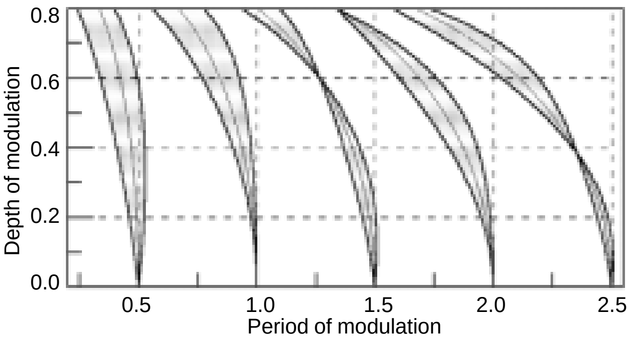

Figure 5. Intervals of parametric excitation at the square-wave modulation of the moment of inertia in the absence of friction.

what greater than $m = \pi/(2Q)$ given by Equation 5 for square-wave modulation, in agreement with the qualitative conclusion I've discussed that the square-wave modulation provides more favorable conditions for the transfer of energy to the oscillator from the source that moves the weights along the rod.

The frequency ranges of parametric excitation

The threshold for the principal parametric resonance (for which $T = T_0/2$) given by Equation 5 is valid for small values of m. For large values of the modulation depth m, the notion of a natural period needs a more precise definition. Let $T_0 = 2\pi/\omega_0 = 2\pi\sqrt{I_0/D}$ be the rotor's period of oscillation when the weights are fixed at some middle positions that correspond to a mean value of the moment of inertia $I_0 = (I_{max} + I_{min})/2$. The period is somewhat longer when the weights are moved further apart: $T_1 = T_0\sqrt{1+m} \approx T_0(1 + m/2)$. The period is shorter when the weights are moved closer to one another: $T_2 = T_0\sqrt{1-m} \approx T_0(1 - m/2)$.

It is convenient to define the average period T_{av} not as the arithmetic mean $(T_1 + T_2)/2$, but rather as the period that corresponds to the arithmetic mean frequency $\omega_{av} = (\omega_1 + \omega_2)/2$, where $\omega_1 = 2\pi/T_1$ and $\omega_2 = 2\pi/T_2$. So we define T_{av} by the relation

$$T_{av} = \frac{2\pi}{\omega_{av}} = \frac{2T_1T_2}{T_1 + T_2}. \qquad (7)$$

The period of modulation T, which is exactly tuned to any of the parametric resonances, is determined not only by the order n of this resonance, but also by the depth of modulation m. To satisfy the resonant conditions, the increment in the phase of natural oscillations during one cycle of modulation must equal $\pi, 2\pi, 3\pi, \ldots, n\pi, \ldots$. During the first half-cycle, the phase increases by $\omega_1 T/2$ and by $\omega_2 T/2$ during the second half-cycle, and instead of the approximate condition of resonance (Equation 2), we obtain

$$\frac{\omega_1 + \omega_2}{2} T = n\pi,$$

$$T = n\frac{\pi}{\omega_{av}} = n\frac{T_{av}}{2}. \qquad (8)$$

Thus, for a parametric resonance of some definite order n, we can express the condition for exact tuning in terms of the harmonic mean period T_{av} of the two natural periods, T_1 and T_2, defined by Equation 7. This simple condition is $T_n = nT_{av}/2$.

An infinite growth of the amplitude during parametric excitation in this linear system is possible not only at exact tuning to one of resonances but also in certain intervals surrounding the resonant values $T = T_{av}/2$, $T = T_{av}$, $T = 3T_{av}/2$, …. Generally, the intervals' width increases with the depth of modulation. Outside the intervals, a torsion pendulum's equilibrium position is stable—that is, the amplitude does not grow and the oscillations damp if there is friction in the system.

When the period of modulation T corresponds to one of the boundaries, the oscillations can be stationary. For the square-wave modulation, we can represent these stationary oscillations as an alternation of free oscillations with the periods T_1 and T_2 occurring during the moment of inertia's intervals of constancy. The graphs of such oscillations are formed by joined segments of sine curves symmetrically truncated on both sides in the absence of friction, and by segments of damped sine curves of natural oscillations otherwise (see Figure 4, whose upper part corresponds to an idealized frictionless system). Without friction, the abrupt increments of the velocity occurring twice during a period are equal to the decrements caused by the modulation of the moment of inertia. With friction, the increments are greater than decrements and compensate for the energy losses caused by friction.

I have derived the boundaries of the intervals of parametric resonance in the user's manual (pp. 110–114)[2] for an idealized frictionless system, and I show an approximate equation valid for the system with relatively weak viscous friction in the instructor's guide (pp. 73–76).[3] Figure 5 shows the intervals of instability that surround the first five parametric resonances as functions of the depth m of the square-wave modulation. The central line of each "tongue" in the diagram shows the period $T = nT_{av}/2$ that corresponds to exact tuning to n-order resonance.

For small values of m, the intervals surrounding resonances of even orders ($n = 2$, 4) are very narrow compared to odd resonances ($n = 1, 3, 5$). To understand why resonances of even orders are so weak and narrow, we should consider that the abrupt changes of the moment of inertia for, say, $n = 2$ resonance, induce both an increase and a decrease of the angular velocity only once during each natural period. The oscillations grow only if the increment of the velocity at the instant when the weights are drawn closer is greater than the decrement occurring when the weights are drawn apart. This is possible only if the weights shift toward the axis when the rotor's angular velocity is greater in magnitude than it is when they shift apart. Such conditions are easily fulfilled for odd resonances because the weights shift apart at extreme points where the velocity is zero. For $T \approx T_{av}$, we can fulfill the

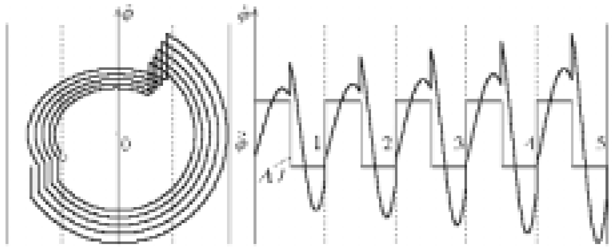

Figure 6. The phase trajectory and the graph of the angular velocity of oscillations corresponding to parametric resonance of the second order ($T = T_{av}$).

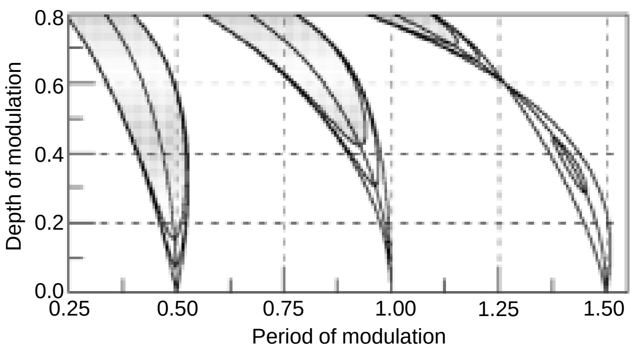

Figure 7. Intervals of parametric excitation at square-wave modulation of the moment of inertia in the absence of friction and at weak friction (for Q = 20 and Q = 10).

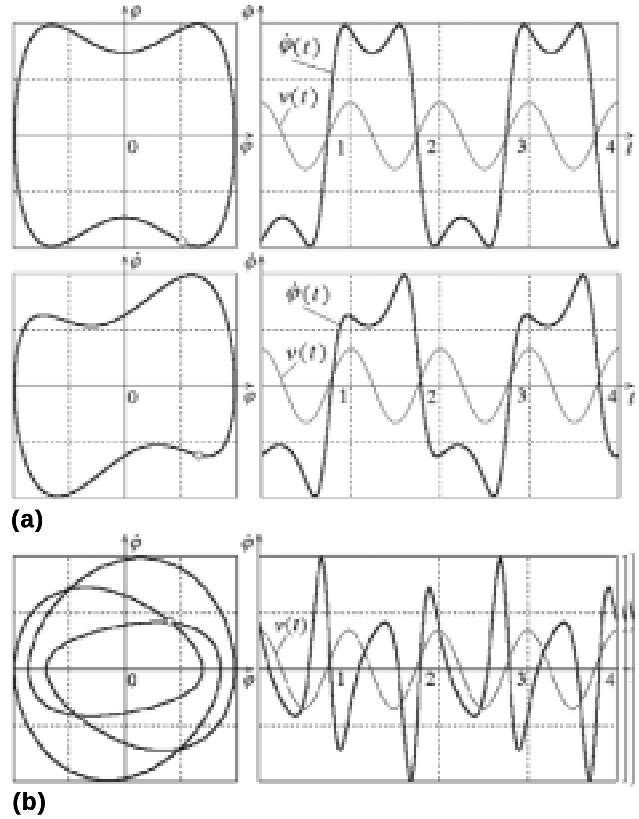

Figure 8. Stationary parametric oscillations: (a) at the upper boundary of the principal interval in the case of sinusoidal modulation; (b) at the threshold of the third parametric resonance.

conditions for oscillation growth only because there is a (small) difference between the rotor's natural periods T_1 and T_2, where T_1 is the period with the weights shifted apart and T_2 is the period with them shifted together. This difference is proportional to the depth of modulation m and vanishes when m tends to zero. Figure 6 shows the growth of oscillations at the second-order parametric resonance.

Figure 5 shows that with the growth of the depth of modulation m, the even intervals expand and become comparable with the intervals of odd orders. For certain values of m, both boundaries of intervals with $n > 2$ coincide (we may consider that they *intersect*). Thus, at these values of m, the corresponding intervals of parametric resonance disappear. These values of m correspond to the natural periods T_1 and T_2 of oscillation (associated with the weights far apart and close to each other) whose ratio is 2:1, 3:1, and 3:2. For the corresponding values of the modulation depth m and the period of modulation T, oscillations are steady for arbitrary initial conditions.

When there is friction in the system, the intervals of instability become narrower, and for strong enough friction (below the threshold), the intervals disappear. Figure 7 shows the boundaries of the first three intervals of parametric resonance in the absence of friction, for Q = 20 and for Q = 10. For any given value m of the depth of modulation, only the first several intervals (if any) of parametric resonance can exist for which m exceeds the threshold. Note the "island" of parametric resonance for n = 3 and Q = 20. This resonance disappears when the depth of modulation exceeds 45% and reappears when m exceeds 66%.

A smooth modulation of the moment of inertia

When the moment of inertia I of the rotor is subjected to a smooth variation, the angular momentum $I\dot{\varphi}(t)$ changes in time according to the equation

$$\frac{d}{dt}\left(I\dot{\varphi}\right) = -D\varphi, \tag{9}$$

where $-D\varphi$ is the restoring torque of the spring. Substituting into Equation 9 $l(t)$ from Equation 6 and taking into account the expression $\omega_0^2 = D/I_0$, where $I_0 = 2Ml_0^2$ is the moment of inertia of the rod (with the weights of mass M each in their mean positions), we obtain

$$\frac{d}{dt}\left[\left(1+\overline{m}\sin\omega t\right)^2\dot{\varphi}\right] = -\omega_0^2\varphi - 2\gamma\dot{\varphi}. \tag{10}$$

We added the drag torque of viscous friction to the right-hand side of Equation 10. The computer program solves this equation in real time during the simulation of oscillations at sinusoidal motion of the weights.[2]

To obtain an approximate solution that is valid up to terms of the first order in the small parameter $\overline{m}$, we can, instead of the exact differential equation of motion (Equation 10), solve the following approximate equation:

$$\ddot{\varphi} + 2\gamma\dot{\varphi} + \omega_0^2\left(1 - 2\overline{m}\sin\omega t\right)\varphi = 0. \qquad (11)$$

Equation 11 is a special case of Hill's equation (Equation 1), with sinusoidal time dependence of the parameter. It is called *Mathieu's equation.* The theory of Mathieu's equation has been fully developed, and all significant properties of its solutions are well-known. A complete mathematical analysis of its solutions is rather complicated and is usually restricted to determining the frequency intervals in which the state of rest in the equilibrium position becomes unstable: At arbitrarily small deviations, the amplitude of incipient small oscillations begins to increase progressively in time. The boundaries of these intervals of instability depend on the depth of modulation $2\overline{m}$.

Even inside the intervals (when conditions for parametric resonance are satisfied), if φ and $\dot{\varphi}$ are exactly zero simultaneously, they remain zero. This property contrasts with the usual case of resonance in which an external force acts upon the system. In forced oscillations, the amplitude begins to grow even from the state of rest in the equilibrium position.

The application of the theory of Mathieu's equation to the simulated system is restricted to the linear order in $\overline{m}$. For finite values of the depth of modulation $\overline{m}$, the resonant frequencies and the boundaries of the intervals of instability for the simulated system differ from those predicted by Mathieu's equation. In the *Physics of Oscillation*'s user manual and instruction guide,[2,3] I present an approximate analytical solution to the exact differential equation of motion (Equation 8), valid up to the terms of the second order in $\overline{m}$ for the main resonance and resonance of the second order, using the method described in *Mechanics*.[4] In particular, for the main resonance, this solution gives the threshold condition that coincides with the condition $m_{\min} = 2/Q$ (where $m \approx 2\overline{m}$), obtained from considerations based on the conservation of energy. For the second resonance, this approximate solution gives the following threshold condition:

$$\overline{m}_{\min} = 2/\sqrt{Q},\ Q_{\min} = 4/\overline{m}^2. \qquad (12)$$

The threshold for the second parametric resonance with smooth modulation is also somewhat greater than with square-wave modulation: compare the expression for $m_{\min}$ given by Equation 12 with $m_{\min} = \sqrt{2/Q}$ (where $m \approx 2\overline{m}$).

The simulation program in the user manual lets us study parametric oscillations and obtain graphs of the variables for arbitrarily large values of the depth of modulation $\overline{m}$. Figure 8a shows an example of steady oscillations occurring at the upper boundary of the principal instability interval (the frequency of modulation $\omega \approx 2\omega_0$). Its upper part corresponds to an idealized frictionless system. The contribution of higher harmonics (mainly of the third harmonic with the frequency $3\omega/2$) cause the shape of the plots to deviate from a sine curve.

For a smooth modulation of the moment of inertia, parametric resonance of the third order is weaker and narrower than for the second order (in contrast to the case of square-wave modulation, for which at $m << 1$ the third-order resonance is stronger and wider than the second-order resonance). This third-order interval disappears in the presence of very small friction. Figure 8b shows stationary oscillations at the threshold of parametric resonance of the third order.

The programs in the *Physics of Oscillations* software are flexible and sophisticated enough for use in student research projects for exploring new properties. Visualizing motion simultaneously with plotting the graphs of different variables and phase trajectories makes the simulation experiments very convincing and comprehensible. These simulations bring to life many abstract concepts of the physics of oscillations and provide a good background for the study of more complicated nonlinear parametric systems such as a pendulum whose length is periodically changed or a pendulum with the suspension point periodically driven vertically.

Under certain conditions, such simple mechanical systems, although described by deterministic laws, exhibit irregular, *chaotic* behavior. Discovery of chaotic motions in simple deterministic dynamical systems of different nature (physical, chemical, biological) is one of the most prominent recent scientific sensations. I plan to include simulations of such systems in the second part of the *Physics of Oscillations.*

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