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Families of Keplerian orbits

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Abstract

Various properties of Keplerian orbits traced by satellites that are launched from one and the same spatial point with different initial velocities are discussed. Two families of elliptical orbits are investigated, namely the sets of orbits produced by a common direction but different magnitudes of the initial velocities, and by a common magnitude but various directions of the initial velocities. For the latter family, the envelope of all the orbits is found, which is the boundary of the spatial region occupied by the orbits.

1. Introduction: Keplerian orbits with common initial point

In most introductory textbooks on mechanics (see, for example, [1]) the study of orbital motion in a central Newtonian gravitational field, e.g. the motion of artificial satellites around the earth, is usually restricted to examples in which the initial velocity of a satellite is directed horizontally. One of such orbits is circular. It is generated by a satellite whose initial velocity v_0 equals the circular velocity for the given initial point: $v_0 = v_{\rm circ} = \sqrt{gR^2/r}$, where g is the acceleration of free fall near the earth's surface, R is the earth's radius, and r is the distance of the initial point from the centre of the earth. The major axes of all other orbits traced by satellites with transversal initial velocities (smaller in magnitude than the escape velocity $v_{\rm esc} = \sqrt{2}v_{\rm circ} = \sqrt{2gR^2/r}$) are oriented along the vertical line passing through the initial position and the centre of the earth. If the initial speed exceeds the circular velocity, the initial point is the perigee (point of the orbit closest to the earth's centre), while the remotest point of the orbit—the apogee—is located at the opposite end of the major axis. Otherwise, the initial point is the apogee, and if the ellipse does not intersect the surface of the earth, the other end of the major axis is the perigee of the orbit. In the case where the ellipse intersects the earth, the outer portion of the ellipse represents the trajectory of a ballistic missile rather than an orbit of a satellite.

The most distinctive feature of motion in the central inverse square gravitational field is a very simple geometrical form of possible trajectories: they all are conic sections for arbitrary initial conditions. In particular, for negative values of the total energy the trajectories are closed curves—ellipses or circles. In this paper we discuss Keplerian orbits that are generated by satellites whose initial velocities at a given initial point have arbitrary directions. Different families of elliptical orbits are characterized by various interesting peculiarities which can be

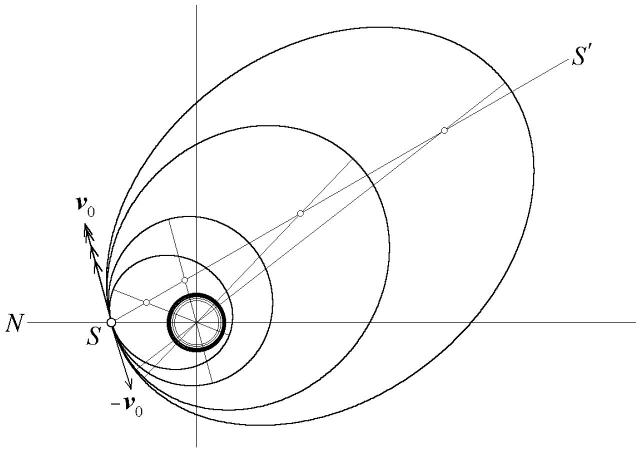

Figure 1. Elliptical trajectories of satellites launched in one direction with different magnitudes of the initial velocities.

derived from simple geometrical properties of the ellipses. The peculiarities of such families of orbits discussed below can be used to solve various problems concerning the orbital motion. The figures included in this paper to illustrate these peculiarities are obtained with the help of the educational software package 'Planets and Satellites' [2] developed by the author.

2. Orbits of satellites launched in one direction with different magnitudes of the initial velocities

We next consider the family of elliptical orbits traced by satellites that are all launched from one and the same spatial point in a central Newtonian gravitational field and in the same direction but at different speeds.

For any non-horizontal direction of the initial velocity, a circular orbit cannot be generated no matter what the magnitude of the initial velocity might be. Several orbits for which the initial velocities v_0 make the same acute angle with the upward vertical are shown in figure 1. The initial position S is the only point common to all the orbits. They are tangent to one another at this point, because their velocity vectors lie in the same direction here. The major axes of the ellipses have different orientations.

The same family of elliptical orbits is generated by satellites launched from the same initial point in the opposite direction: with the initial velocity $-\boldsymbol{v}_0$ (see figure 1) a satellite will move oppositely along exactly the same ellipse as the satellite with the initial velocity $\boldsymbol{v}_0$.

An interesting property of this collection of elliptical orbits is related to the position of their foci. One focus is common to all the orbits. It is located at the centre of the earth and so of course this set of foci is represented by a single point. The foci of the other set are located on one and the same straight line SS' that passes through the initial position S (see figure 1). This line SS' forms an angle with the upward vertical SN at the initial position that is twice the angle formed by the initial velocity with the upward vertical. (For horizontal initial velocities, this property means that the foci of this second set lie on a downward vertical through the initial position.)

The property mentioned above of the family of orbits under consideration can be easily proved geometrically if we remember the optical property of the ellipse, according to which all light rays emanating from one focus are reflected by an elliptical mirror toward the other focus. Let us imagine a ray that is emitted from the centre of the earth towards point S and is reflected here towards S' by a plane mirror that is tangent to all the ellipses. This reflected ray

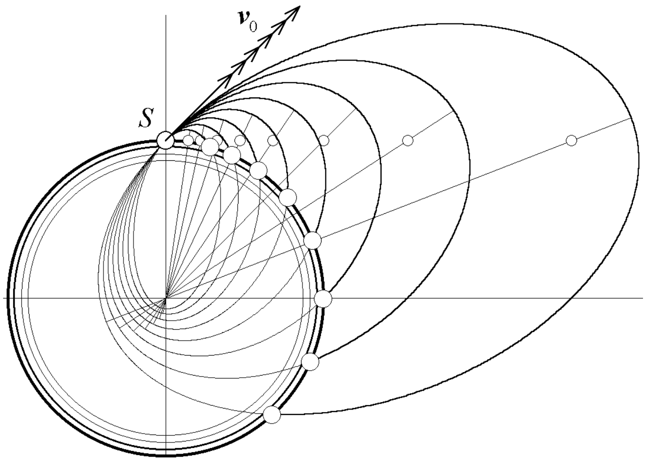

Figure 2. Ballistic trajectories of missiles launched at an angle of 45° with different magnitudes of the initial velocities.

SS' passes through the second foci of all the ellipses. The angle of reflection at point S equals the incident angle. This means that the straight line SS' on which these foci are located makes an angle with the vector $\boldsymbol{v_0}$ that equals the angle between the line from the centre of the earth towards S and the vector $-\boldsymbol{v_0}$. In other words, the vector $\boldsymbol{v_0}$ is directed along the bissectrix of the angle NSS' (see figure 1).

Figure 2 shows a set of ballistic trajectories of missiles launched at different speeds from a point on the earth's surface. The velocity of each is directed at an angle of 45° to the upward vertical. In the absence of air resistance, these trajectories are parts of ellipses with a common focus at the centre of the earth. If the initial velocity is small compared to the circular velocity, the portion of the ellipse above ground is approximately a parabola. This is just the parabolic trajectory that we usually assign to a projectile in the approximation of a 'flat earth' and in the absence of air resistance.

For small initial velocities and consequently short ranges (compared to the earth's radius), the gravitational field of the earth can be considered uniform, i.e. constant in magnitude and direction (homogeneous) along the whole trajectory, and the above approximation is clearly applicable. However, we should keep in mind that actually the trajectory is a portion of an ellipse with one of the foci at the centre of the earth.

All the trajectories of the set are tangent at the initial position. Orientations of their major axes depend on the magnitude of the initial velocity. We note that the other foci of all the ellipses are located on the straight line that passes horizontally through the initial position. As we have shown above, such an alignment of the second foci is explained by the optical property of the ellipse. The projectile whose initial velocity equals in magnitude the circular velocity (called also the first cosmic velocity) hits the ground at a point whose angular distance from the initial position is exactly 90°. The major axis of this orbit with $v_0 = v_{\text{circ}}$ is parallel to the initial velocity.

3. Satellites with equal magnitudes of the initial velocities

Imagine a rocket, launched from the earth, rising vertically, and at the highest point of its flight exploding into many identical fragments that fly off in all directions with equal speeds. The further motion of the fragments occurs under the action of the central force of the earth's gravity. Thus, they become earth's satellites, orbiting along various elliptical Keplerian orbits. Several orbits of such families are shown in figures 3 and 4. For the family of orbits in figure 3

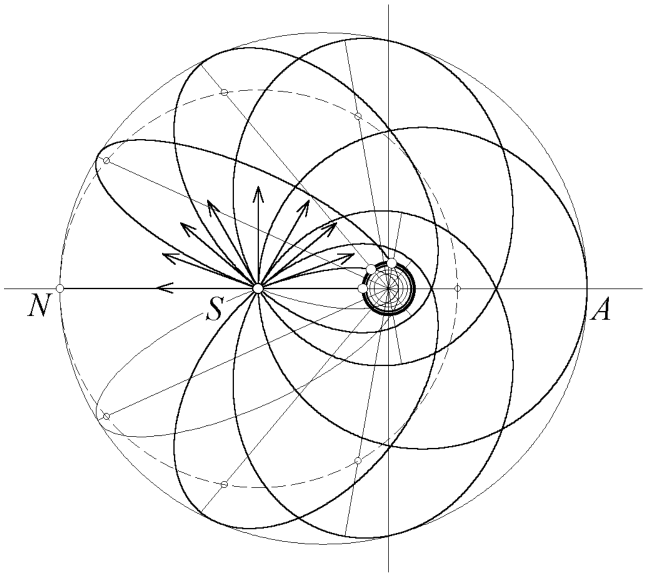

Figure 3. A family of elliptical orbits of fragments scattered from the same initial position S in different directions with equal speeds. This speed exceeds that which generates a circular orbit ($v_0 > v_{\mathrm{circ}}$).

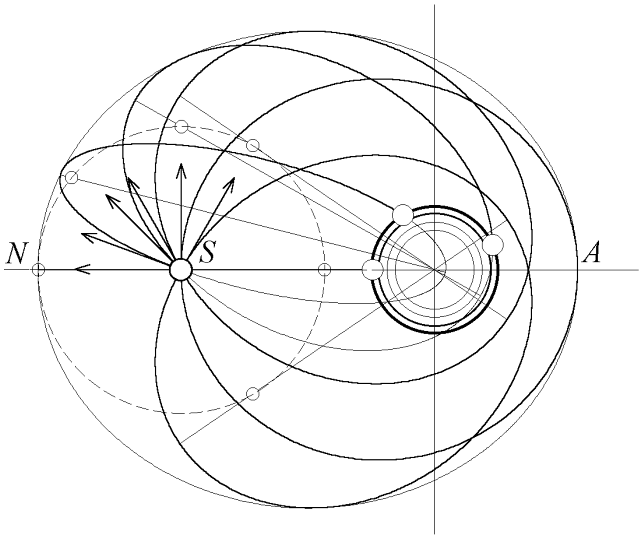

Figure 4. Elliptical orbits and ballistic trajectories of fragments scattered from one initial position S for the case in which $v_0 < v_{\mathrm{circ}}$.

the initial velocity v_0 of the satellites is greater in magnitude than the circular velocity v_{circ} for the initial position S, and in figure 4, smaller.

For a given family, one of the foci is common to each of the elliptical orbits. It is located at the centre of the earth. The second focus of each orbit lies on a circle whose centre is located at the common initial position S. This circle is shown by a dashed line in figures 3 and 4. Its radius equals the distance between the initial position S and the highest point N reached by

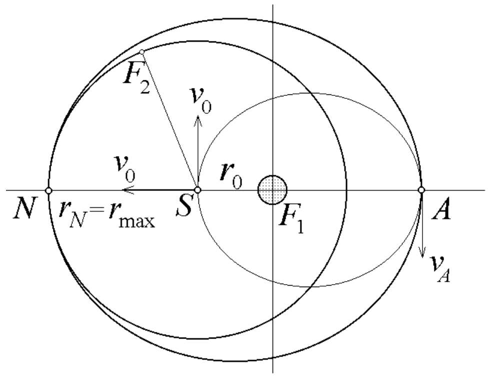

Figure 5. For the geometric determination of the boundary surface.

the fragment that moves vertically upward from the initial position S. Since the magnitudes of the initial velocities of each of the fragments are equal, and since the motion of each fragment begins at the same spatial point, the total energies of each are equal. Therefore, the major axes of the orbits are the same, and by Kepler's third law, so also are the periods of revolution. That is, all the fragments whose elliptical orbits do not intersect the earth's surface, simultaneously return to the initial position after a revolution.

These orbits are confined to a particular spatial region. Its boundary is an axially symmetric surface whose axis of symmetry passes through the centre of the earth and the point at which the explosion occurs. A detailed analysis (see the next section) shows that it is a surface of revolution of an ellipse whose foci are at the centre of the earth and the initial position S. Diagrams of this ellipsoid are shown in figures 3 and 4 in which the enveloping surface (more precisely, its section by a plane) is depicted by a thin solid curve. The dimensions and eccentricity of the ellipsoid are determined by the position of the initial point and by the magnitude of the initial velocities of the fragments. Next we give a rigorous derivation of the parameters of this bounding surface (based on the geometric properties of Keplerian orbits). For the limiting case of small initial speed of the fragments (much smaller than the circular velocity: $v_0 \ll v_{\text{circ}} = \sqrt{gR^2/r}$) this bounding surface shrinks into a very narrow (degenerate) ellipsoid spanned across the foci located at the initial point S and the centre of the earth. The portion of this ellipsoid that covers the starting point can be approximated by the commonly known envelope of the parabolic trajectories of the projectiles in the uniform gravitational field fired at a certain speed in various directions. The envelope surface in this limiting case is a paraboloid of rotation about the vertical line passing through the initial point (see for example [3]).

4. The envelope of orbits with equal energies

Below we suggest a simple geometrical proof that the bounding surface for the family of elliptical orbits with equal energies and a common initial point is an ellipsoid, generated by the rotation of an ellipse about the axis of symmetry NA (see figures 3 and 4). One of the foci of the ellipse is located at the centre of the earth and the other at the initial point. The dimensions and eccentricity of the ellipsoid depend on the position of the initial point S and on the initial velocity v_0 of the fragments.

The fragment whose initial velocity, at the point S in figure 5, is directed upward along the local vertical line (to the left in figure 5), rises vertically along a straight line to the highest point N located at the distance r_{max} from the earth's centre. Then it falls toward the earth along the same line. The trajectory of this fragment is part of the rectilinear segment joining the highest point N with the centre of the earth F_1. We can consider this segment as the limiting case of an infinitely narrow ellipse. The foci of this degenerate ellipse lie at the ends of the segment. That is, one focus is at the earth's centre F_1 and the other at the highest point N of the trajectory.

Clearly the highest point N is on the bounding surface. The distance $r_{\max} = r_N$ of this point from the earth's centre can be calculated by equating the total energy of the fragment at this point N to the total energy at the initial point S, located at the distance r_0 from the centre of force:

$$\frac{v_0^2}{2} - \frac{GM}{r_0} = -\frac{GM}{r_{\max}}. \tag{1}$$

It is convenient to express in equation (1) the gravitational parameter of the planet GM in terms of the escape velocity v_{esc} for the initial point S ($v_{\text{esc}}^2 = 2GM/r_0$):

$$\frac{1}{r_{\max}} = \frac{1}{r_0}\left(1 - \frac{v_0^2}{v_{\text{esc}}^2}\right); \qquad r_{\max} = r_N = \frac{r_0}{1 - (v_0/v_{\text{esc}})^2}. \tag{2}$$

If the initial velocity equals the circular velocity for the initial point, that is, if $v_0 = v_{\text{circ}} = \sqrt{GM/r_0}$, equation (2) gives $r_{\max} = r_N = 2r_0$: the distance of the highest point N from the earth's centre is twice the distance r_0 of the initial point.

We can easily find one more point on the boundary, namely, the point opposite N. It coincides with the apogee A (or with the perigee if $v_0 < v_{\text{circ}}$) of the elliptical orbit of that fragment whose initial velocity at S is directed horizontally (transverse to the radius vector). The distance r_A of this point from the earth's centre can be calculated with the help of the laws of energy conservation and angular momentum conservation ($r_0 v_0 = r_A v_A$):

$$r_A = \frac{r_0}{(v_{\text{esc}}/v_0)^2 - 1}. \tag{3}$$

Next we prove that the curve whose rotation generates the boundary is an ellipse. The ends of the major axis of this ellipse are located at N and A, and its foci are located at the initial point S and the earth's centre F_1. The shape of the boundary is determined from the following geometric properties of the trajectories.

First we find the locus of the foci of all the orbits of the fragments. All orbits have a focus at the centre of the earth, and so the locus of this set is the point F_1. The locus of the set of second foci F_2 is a circle whose centre is located at the initial point S, and whose radius is equal to the distance $|SN|$, measured from S to the most remote point N (see figure 5). Indeed, for any orbit of the family, the sum of two distances of each point on the orbit from the foci of the orbit equals the major axis of the orbit. The major axes of all the orbits of the family, as we have seen, are equal to each other. Their lengths are equal to the length $r_{\max}$ of the segment F_1N. This segment can be considered the major axis of the degenerate elliptical orbit of the fragment whose initial velocity is directed upwards. All orbits of the family pass through S, and the distance from this common point to the focus F_1 for all the orbits equals r_0. Consequently, the distance between S and the second focus also must be equal for all the orbits. Hence the second foci of all orbits of the family lie on the circle with the centre at the initial point S and radius $|SN|$.

Next we consider the following auxiliary construction (figure 6): we draw a second circle with the centre at the earth's centre F_1 and radius $r_{\max} = |F_1N|$. This circle passes through N, which lies on the bounding surface.

Now let us consider the problem of finding the orbit passing through an arbitrary point M that lies within the second circle just drawn. Choosing M as a centre, we draw a third circle tangent to the second circle at point B, as shown in figure 6. The second focus of the desired orbit passing through M must lie on this circle because the sum of two distances from the foci again must be equal to $r_{\max}$. And at the same time the second focus must lie on the first circle (with the centre at the initial point S and radius $|SN|$). We examine three possibilities:

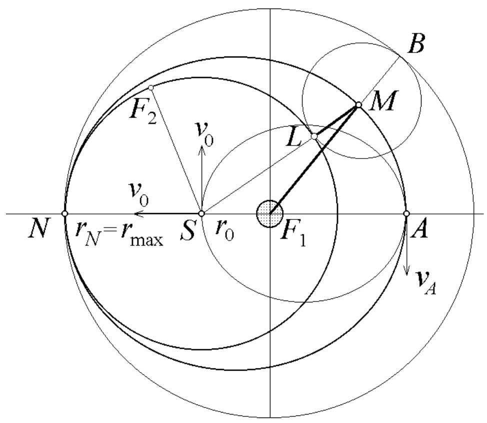

Figure 6. For the geometrical proof of elliptical shape of the bounding surface (see text for detail).

(1) The third circle (with the centre at M) *intersects* the first circle (the locus of second foci of the orbits) at two points. Then there exist *two orbits* of the family that pass through the given point M. The second foci of these two orbits lie at the two points of intersection.
(2) The third circle has *no common points* with the first circle. Then no orbit of the family passes through M. It follows that M lies *outside* the bounding surface.
(3) Lastly, the third circle *grazes* the first circle, thus having a single common point L with it (see figure 6). Then *only one orbit* of the family passes through M. In this case point M must lie *on* the bounding surface. At point M this single orbit grazes the envelope surface. The foci of this orbit are located at F_1 and L.

We can see from figure 6 that in the latter case of grazing the sum of distances from M to F_1 and S equals the radius r_{max} of the second circle plus the radius $|SN|$ of the first circle. This sum is independent of the position of point M on the boundary. That is, the sum has equal values for all points of the boundary. Since points F_1 and S are fixed, and since the sum of their distances from M is the same for all M, we have proved that the locus of the boundary points for the region occupied by the orbits of the family is an ellipse whose foci are at F_1 and S.

We note that the exploitation of geometrical properties of ellipses allowed us to easily find the envelope of the family of orbits without tedious calculations.

The eccentricity of the bounding ellipse can be found as the ratio of the distance r_0 between its foci to the major axis $r_N + r_A$. Using equations (2) and (3), we find

$$e = \frac{r_0}{r_N + r_A} = \frac{v_{esc}^2 - v_0^2}{v_{esc}^2 + v_0^2}. \tag{4}$$

If we increase the initial speed of the fragments, the bounding surface expands and its shape becomes almost spherical. Indeed, as we can see from equation (4), its eccentricity e becomes smaller and tends to zero as the initial speed approaches the escape velocity. If the initial velocity v_0 equals the circular velocity v_{circ} for the initial point, the distance between the foci of the bounding ellipse is one-third its major axis. That is, the eccentricity of the ellipse is $1/3$ if $v_0 = v_{circ}$. If the initial speed tends to zero, the eccentricity of the envelope approaches unity. The apexes (the ends of the major axis) of the bounding ellipse approach its foci, and the ellipse becomes very narrow and stretched, being spanned over the initial point S and the centre of the earth F_1. This limiting case of a degenerate ellipse corresponds to the parabolic shape mentioned above (see [3]) of the bounding surface for the trajectories of the fragments moving within a restricted spatial region (in the vicinity of the initial point S) in which the gravitational field can be regarded as uniform.

5. Applications of the envelope

Knowing the boundary surface can be very useful in solving various problems concerning the orbital motion. For example, we can easily find the minimal firing speed of a projectile needed to hit a given target from a given starting point. Suppose we have a target M (see figure 6) at a given location, which is determined by distance $r_M = |F_1M|$ from the force centre F_1 (the centre of the earth) and distance $l_M = |SM|$ from the given starting point S (whose distance from the force centre is $r_0 = |F_1S|$). What is the minimal initial speed and what should be the firing angle?

Obviously, the firing speed is minimal if the target M lies on the bounding surface. Since this boundary is an ellipse, the sum of distances $|F_1M|$ and $|SM|$ from M to its foci (points F_1 and S) is equal to the major axis of the bounding ellipse: $|F_1M| + |SM| = r_N + r_A$. The sum $|F_1M| + |SM|$ is just the sum of given distances r_M and l_M to the target from F_1 and S. Let us denote this sum as b: $r_M + l_M = b$. Thus, we can equate this given value b to the major axis $r_N + r_A$ which has already been calculated above, when we derived equation (4) for the eccentricity of the bounding ellipse:

$$b = r_N + r_A = r_0 \frac{v_{\mathrm{esc}}^2 + v_0^2}{v_{\mathrm{esc}}^2 - v_0^2}. \tag{5}$$

Solving equation (5) for v_0, we obtain the desired minimal firing speed:

$$v_{0\ \mathrm{min}}^2 = v_{\mathrm{esc}}^2 \frac{b - r_0}{b + r_0}. \tag{6}$$

Equation (6) shows that for a given position of the starting point S the minimal firing speed depends only on b, that is, on the sum of distances r_M and l_M that determine the target location ($b = r_M + l_M$). According to equation (6), the firing speed is zero if $b = r_0$, that is, for any target that lies on the segment SF_1 joining the starting point and the centre of the earth. The minimal firing speed v_0 tends to the escape velocity $v_{\mathrm{esc}} = \sqrt{2gR^2/r}$ as the target is moved away to infinity (as $b \to \infty$).

The trajectory of the projectile fired to the given target M with the minimal starting speed is a portion of the ellipse passing through S and M. One focus of this Keplerian ellipse is located at the centre of the earth, while the second focus belongs to the circle whose centre is at the starting point S and whose radius equals $|SN|$ (see figure 6). Therefore this second focus is located at point L at which the segment SM from the starting point to the target intersects the mentioned circle. Knowing locations of both foci for the elliptical trajectory of the projectile, we can easily find the firing angle with the help of the optical property of the ellipse. Since the light ray emitted from the focus F_1 to S must be reflected at S by the elliptical mirror towards the second focus L, the tangent to the ellipse at S (and hence the direction of initial velocity $\boldsymbol{v}_{0\ \mathrm{min}}$) is the bissectrix of the angle LSN (or MSN, see figure 6).

Particular examples of trajectories traced by the projectiles launched with minimal initial speeds to given targets are shown in figure 7. Let the starting point S be located at the height of one-third the earth's radius R over the North Pole, and the target lie on the equator (figure 7(a)). Therefore $r_0 = \frac{4}{3}R$, $r_M = R$, and $l_M = \frac{5}{3}R$, so that $b = \frac{8}{3}R$. For the minimal starting speed in this case equation (6) yields $v_{0\ \mathrm{min}}^2 = \frac{2}{3}v_{\mathrm{circ}}^2$, $v_{0\ \mathrm{min}} = 0.8165\, v_{\mathrm{circ}}$. From the triangle F_1SM we can see that the sine of angle MSN equals $\frac{3}{5}$. The initial velocity $\boldsymbol{v}_0$ must be directed along the bissectrix of this angle. Hence the angle between vector $\boldsymbol{v}_0$ and the upward vertical line must equal 71.565°. The trajectory of this projectile is a portion SM of an ellipse whose foci are located at F_1 (the earth's centre) and F_2. The latter point lies on the straight line SM joining the starting point and the target.

We note that at the target point M (see figure 7) both ellipses (the trajectory and the envelope bounding surface) have common tangent. According to the optical property, the ray F_1M emitted from the common focus F_1 of these ellipses must be reflected at M by both

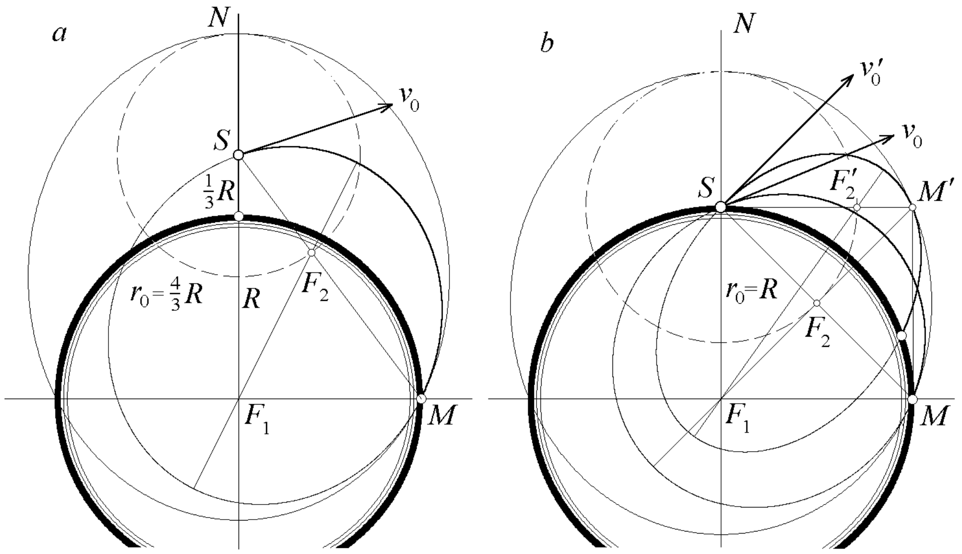

Figure 7. Examples of trajectories that correspond to the minimal starting speed of the projectile for a given starting point S and a given target M.

curves toward their second foci (F_2 and S respectively). Therefore all three points (M, F_2, and S) lie on the same straight line MS.

If the starting point S is located on the surface of the earth (figure 7(b)), for the same target M on the equator we have $l_M = \sqrt{2}R$, $b = (1 + \sqrt{2})R$, and equation (6) yields $v_{0\,\mathrm{min}} = 0.9102\, v_{\mathrm{circ}}$. The angle MSN in this case equals 135°, so that vector $\boldsymbol{v_0}$ must make an angle of 67.5° with the upward vertical.

The same minimal starting speed is required for the target M' located in space at the distance R on the straight line directed horizontally from point S (see figure 7(b)). We can make this conclusion either analytically from equation (6), or geometrically from the observation that both targets M and M' belong to the same bounding surface. To hit this target, the vector $\boldsymbol{v_0}$ must make an angle of 45° with the upward vertical. We note that on the surface of the earth the range of this projectile is smaller than in the preceding case in which the projectile is fired with the same starting speed at an angle of 67.5°.

6. Concluding remarks

In this paper we have tried to show that interesting and useful peculiarities of the families of Keplerian orbits can be obtained by very modest means based solely on the fact that these orbits are ellipses characterized by commonly known simple geometrical properties. Many demanding problems concerning the orbital motion can be easily solved with the help of these properties. Simple geometrical considerations can provide elegant solutions which allow us to avoid complicated and tedious calculations.

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