

# Relative motion of orbiting bodies

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**Abstract.** A problem of relative motion of orbiting bodies is investigated on the example of the free motion of any body ejected from the orbital station that stays in a circular orbit around the earth. An elementary approach is illustrated by a simulation computer program and supported by a mathematical treatment based on approximate differential equations of the relative orbital motion.

## 1. Relative motion of bodies in space orbits—an introductory approach

Let two satellites orbit the earth. We know that their passive orbital motion obeys Kepler's laws. But how does one of them move relative to the other? This relative motion is important, say, for the spacecraft docking in orbit. If two satellites are brought together but have a (small) nonzero relative velocity, they will drift apart nonrectilinearly. In unusual conditions of the orbital flight, navigation is quite different from what we are used to here on the earth, and our intuition fails us. The study of the relative motion of the spacecraft reveals many extraordinary features that are hard to reconcile with common sense and our everyday experience.

In this paper we discuss the problem of passive relative motion of orbiting bodies on the specific example of the free motion of any body that is ejected from the orbital station that stays in a circular orbit.

The free motion of an astronaut in the vicinity of an orbiting spacecraft has been investigated in [1]. The discussion in [1] is restricted to a low relative speed and short elapsed time (constituting a small fraction of the orbital period). In our treatment of the problem this time can be as long as several periods. The only restriction concerns the distance between the orbiting bodies, which must be small compared with the dimensions of the orbit. A similar problem has been considered in [2].

It is essential for the problem under consideration that the initial velocity of the body relative to the station be small compared to the orbital velocity of the station. For example, what does the motion of a body thrown down towards the earth look like for the astronauts on the orbital station?

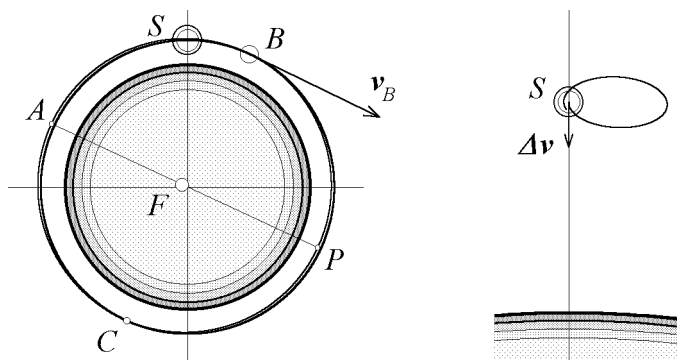
Our thinking about this problem may pass through several stages.

The first stage is likely to be governed by jumping too quickly to a conclusion: Relying on our everyday experience, we find nothing strange if the body thrown toward the earth simply falls rapidly toward the ground.

The second stage begins with a reflection that the orbital station travels over the earth at a great speed—more than seven kilometers per second! What is the initial velocity of the body that is thrown from the moving station by an astronaut?

A healthy man can throw a stone with a speed of about 10–20 m/s. Considering the motion of the body relative to the earth, we should add vectorially the velocity of the orbital station to the velocity of the body with respect to the station. We see that the resulting velocity of the body differs only negligibly in magnitude and direction from the velocity of the station. This means that after being thrown, the body simply transfers to another orbit around the earth, and this new orbit is almost indistinguishable from the orbit of the station. Does this conclusion agree with the first conclusion, that the body rapidly recedes from the station toward the earth?

Next our speculations pass over to the third, exploratory stage. We recall reasonably that the question is concerned not with the motion of the body relative to the earth, but rather with the motion as it is seen by the astronauts on the station. In other words, we should investigate the motion of the body in the frame of reference associated with the orbital station. The simulation program [3] developed recently by the author allows us to observe this relative motion on the computer screen. The left-hand part of Figure 1 shows the circular orbit of the station  $S$ . At point  $B$  the body is ejected downward with an additional velocity  $\Delta\mathbf{v}$ .



**Figure 1.** Motion of the body thrown from the orbital station down toward the earth, as this motion is seen from the earth (left) and from the station (right).

In the right-hand side of the figure we see that relative to the station the body indeed moves at first downward, in the direction of the additional velocity  $\Delta\mathbf{v}$ . However, soon the trajectory turns forward, then upward and backward, and finally, strange as it may seem, the body returns to the station from the opposite side (from above), tracing an almost closed trajectory!

To explain the physical reasons of this strange relative motion of the orbiting bodies, we should consider first the motion of the body and the station relative to the earth. This motion is shown in the left-hand side of Figure 1. Because of the additional velocity directed towards the center of the earth which the body received at point  $B$ , the ejected body transfers to an elliptical orbit of a very small eccentricity. One of the foci is located at the center of the earth, and the second—at a point  $F'$  that is very close to the center. This ellipse almost merges with the circular orbit of the station. We can barely see that only near perigee  $P$  (Figure 1) is the ellipse slightly inside the circle. Near apogee  $A$  it is slightly outside the circle. With great precision we can treat this ellipse as a circle of the same radius, whose center is displaced from the center of the earth towards  $F'$  through the distance half way to  $F'$ .

Since the diameter of the circle and the major axis of the ellipse are almost equal,

the periods of revolution for the station and for the body, according to Kepler's third law, are nearly equal. Both trajectories intersect one another at the initial point  $B$  and at the opposite point  $C$ . At this point, the body is again at the same height as is the station. The station comes to this point  $C$  after exactly half the period of its uniform rotation around the earth.

However, the motion of the body along its elliptical orbit is slightly non-uniform: in accordance with Kepler's second law, the body comes to point  $C$  a bit earlier than the station, since at the middle of this half of its orbit the body passes through the perigee  $P$ , where its speed is slightly greater than that of the station. As a result, at the common point  $C$  of both orbits the body is in front of the station. At this moment the distance between the body and the station reaches a maximum.

During the second half of the revolution, the body passes through the apogee  $A$  of its elliptical orbit. Because its speed decreases slightly in this part of the orbit, the body comes to the common initial position almost simultaneously with the station, approaching it from above. Hence, the motion of the body relative to the station occurs along an almost closed trajectory. One cycle of this motion is completed during a period of revolution of the station in its circular orbit.

Are the astronauts on the orbital station really able to observe this periodic motion of the body? Here we get over to the fourth stage of our investigation—to calculations. Let us next evaluate the linear dimensions of the almost closed loop covered by the body in its motion relative to the station (see the right-hand side of Figure 1).

As we already mentioned, the geocentric elliptical orbit of the body can be approximated by a circle whose center is displaced from the center of the earth along the major axis  $PA$  through the half-distance between the foci. We can easily evaluate this displacement  $\Delta x$ , taking into account that the vectors of velocities for both orbits at the point of their intersection  $B$  are perpendicular to the corresponding radii. (Neither these vectors nor the radii are shown in Figure 1, since the angle  $\Delta v/v_c$  between them is so small that they merge on the image.) Consequently, the displacement of the center equals the radius  $r_0$  of the circle times the angle  $\Delta v/v_c$  between the velocities of the body and the station at point  $B$  (or at point  $C$ ):  $\Delta x = r_0(\Delta v/v_c)$ .

The divergence between the two orbits is greatest near points  $P$  and  $A$  (Figure 1) and just equals the displacement  $\Delta x$  of the center calculated above. Hence, the lowest point  $P$  of the elliptical orbit is lower than the circular orbit of the station by the distance  $\Delta x$ , and the highest point  $A$  is higher by the same distance  $\Delta x$ . Consequently, the vertical size of the loop in the right-hand side of Figure 1 (the minor diameter of the relative trajectory) equals  $2\Delta x = 2r_0(\Delta v/v_c)$ .

We are now ready to make a numerical estimate of a typical size of the trajectory that the body traces relative to the orbital station. Let, for instance, the height of the circular orbit of the station be  $h = 0.1 R \approx 640$  km (radius of the orbit  $r_0 \approx 7$  thousand kilometers, period of revolution  $T \approx 98$  minutes), and the relative initial velocity of the body be  $\Delta v = 15$  m/s, that equals approximately 0.2% of the orbital velocity  $v_c = 7.5$  km/s. In this case the minor diameter of the relative trajectory equals  $2\Delta x = 2r_0(\Delta v/v_c) \approx 28$  km. This value gives an idea of the actual sizes of the loop in Figure 1.

It is unlikely that the astronauts would be able to see a small body at a distance of more than a kilometer. Therefore they can watch it only during the first part of its nearly closed trajectory. Most likely, they lose sight long before the deviation from its

rectilinear downward motion becomes apparent. Thus, until the astronauts lose sight of the body, it simply falls down towards the ground, moving in the direction of the relative initial velocity imparted to the body by the astronaut!

It is also interesting to calculate the size of the relative trajectory in the horizontal direction. As mentioned above, the major diameter of the loop equals the lag of the station behind the body near the common point  $C$  (see the left-hand side of Figure 1) of their orbits. With the help of Kepler's second law, we can calculate the difference  $\Delta T$  of time intervals needed for the station and for the body to cover the half-orbit between common points  $B$  and  $C$ .

For the station, the area  $A$  swept out by the radius-vector in its rotation from  $B$  to  $C$  during the time  $T/2$  is exactly half a circle:  $A = \pi r_0^2/2$ . For the body, whose nearly circular orbit is displaced from the earth's center through  $\Delta x$ , the corresponding area is smaller by  $\Delta A = 2r_0\Delta x = 2r_0^2\Delta v/v_c$ . Hence  $\Delta T/(T/2) = \Delta A/A = 4(\Delta v/v_c)/\pi$ , and  $\Delta T = 4(r_0/v_c)(\Delta v/v_c)$  (since  $T = 2\pi r_0/v_c$ ). Thus, for the maximal lag  $l_{\max}$  of the station behind the body at point  $C$  we obtain:

$$l_{\max} = v_c\Delta T = 4r_0\frac{\Delta v}{v_c}. \quad (1)$$

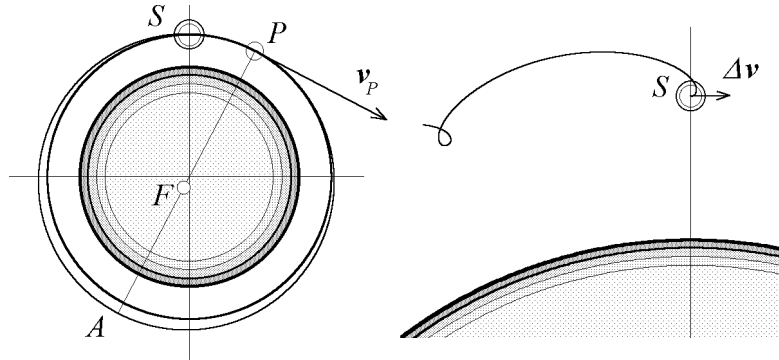
Consequently, the major diameter of the almost closed trajectory of the body in its motion relative to the station is twice its minor diameter. For the values that we used above (the height of the station  $h = 0.1R \approx 640$  km, the relative initial velocity of the body  $\Delta v = 15$  m/s, that equals approximately 0.2% of the orbital velocity  $v_c = 7.5$  km/s), the maximal distance  $l_{\max}$  of the body from the station is approximately 56 km.

Next section contains a detailed derivation of the approximate differential equations that describe analytically the relative motion of a body in the vicinity of the orbital station. For the given initial conditions (a small relative velocity directed downward), the particular solution of the equations predicts the relative motion to be along an elliptical loop stretched horizontally. The semiminor axis  $l$  of the ellipse equals the radius of the circular orbit  $r_0$  times the ratio of the additional velocity  $\Delta v$  to the orbital velocity  $v_c$ , namely  $l = r_0(\Delta v/v_c)$ . The semimajor axis of the ellipse is twice as long. This result is exactly consistent with the estimate of the vertical size of the relative orbit considered above, as well as with the estimate of maximal distance  $l_{\max}$  between the body and the station, given by Eq. (1).

The character of the relative motion of the body thrown from the orbital station is quite different if the relative initial velocity has a component parallel to the orbital velocity of the station. The trajectory of the relative motion is no longer a closed curve even for very small values of the initial relative velocity. The body does not return to the station. In the relative motion, along with the periodic components, there is a non-periodic *secular term*, responsible for the regular, steady receding of the body from the station.

Figure 2 illustrates the trajectory of the relative motion of the body that is thrown forward from the orbital station. At first the body actually moves forward, but gradually it deviates upward, and soon its motion relative to the station becomes retrograde. The body next descends, and the trajectory makes a loop. The body periodically returns to the same height as the station, but with each return lags more and more behind the station.

The general character of this relative motion is easily explained with the help of the geocentric frame of reference. The new orbit of the body is an ellipse grazing the



**Figure 2.** Motion of the body thrown from the orbital station in the direction of the orbital motion, in the frames of reference of the earth (left) and of the orbital station (right).

circular orbit of the station only at the initial point  $P$ , which is the perigee of the elliptical orbit. Passing through the apogee  $A$ , the body is higher than the station. Approaching the perigee  $P$ , the body descends to its initial altitude. But the period of revolution along the ellipse is greater than the period of the station. Therefore after a revolution the body arrives at the only common point  $P$  of the two orbits later than does the station. This lag increases with each revolution. If the initial velocity of the body is such that the ratio of its period to that of the station is rational, the accumulated lag sooner or later becomes equal to the length of the whole orbit (or to several lengths of the orbit), and the body periodically meets with the station.

Another way of understanding the difference in character of the motion in the two cases (the additional velocity either along or perpendicularly to the orbital velocity) comes from the recalling that, for an inverse square force, viewed from the reference frame associated with the source of the force, all orbits with negative energies are closed, and also that the period  $T$  is uniquely related to the semimajor axis  $a$  and hence to the total energy  $E$  – specifically,  $T \propto a^{3/2} \propto (-E)^{-3/2}$ . No matter what the direction and magnitude of the velocity imparted to the ejected body, it will periodically return to and pass through the point in space from which it was thrown. But how will its period compare with that of the station? Throwing the object obviously has no effect on its initial potential energy. If it is thrown in a direction perpendicular to the orbital velocity of the station, then to the first order in  $\Delta v/v_c$  its speed with respect to the geocentric frame will be unchanged, so its total energy (and likewise its period) will remain nearly identical to that of the station. That is, the time at which it returns to the spatial point from which it was thrown will be almost the same as the time of return of the station itself. If, however, its imparted velocity is directed along that of the station (no matter forward or backward), its speed, and hence its total energy and period, will suffer a first-order change.

Indeed, for a circular orbit in the inverse-square central field, the (negative) total energy equals in magnitude the kinetic energy ( $E = -E_{\text{kin}}$ ) and hence is proportional to the square of velocity:  $-E \propto v^2$ . Throwing the body in a direction orthogonal to the station's velocity will produce a second-order fractional change in the energy  $\Delta E/(-E) = (\Delta v/v_c)^2$ . Since  $T \propto (-E)^{-3/2}$ , this will result in the following fractional

change in the period:  $\Delta T/T = (3/2)(\Delta v/v_c)^2$ . For the numbers used above ( $\Delta v = 15$  m/s,  $v_c = 7.5$  km/s), we find  $\Delta T/T = 6 \cdot 10^{-6}$ , which means that when the body will return to the starting point after a period, the station will be ahead of it in the orbit through a distance of about  $v_c \Delta T \approx 265$  m. (We remind that in this motion, after a half-period  $T/2$ , the body occurs at a maximal distance of 56 km from the station.) On the other hand, throwing the body forward or backward produces a first-order fractional change in the energy  $\Delta E/(-E) = 2(\Delta v/v_c)$ , thus resulting in the fractional change of the same order in the period:  $\Delta T/T = 3(\Delta v/v_c) = 6 \cdot 10^{-3}$ . Therefore in this case the station occurs after a period in front of the body through the distance of  $v_c \Delta T \approx 265$  km.

For small values of the initial relative velocity, it is convenient to analyze the motion with the help of approximate differential equations that describe the relative motion of the body in the vicinity of the orbital station. The next section includes a derivation of the equations and their particular solution for the case under consideration.

## 2. A mathematical description of the relative orbital motion

To investigate the relative motion of orbiting bodies quantitatively, we can derive simple linearized differential equations that are approximately valid for small spatial distances between the bodies. More precisely, these equations describe the relative motion of the orbiting bodies while the distances between them are much smaller than the linear dimensions (axes) of the orbit.

As an example, we consider the motion of a body ejected by an astronaut from an orbital station that orbits the earth in a circle. We make use of the non-inertial frame of reference whose origin lies in the station (Figure 3). The  $z$ -axis of this frame points perpendicularly to the plane of the orbit, that is, parallel to the vector of the angular velocity  $\boldsymbol{\Omega}$  of revolution of the station; the  $x$ -axis lies in the plane of the orbit and extends radially outward, away from the center of the earth; and the  $y$ -axis is parallel to the orbital velocity,  $\mathbf{v}_c$ . Vector  $\mathbf{r}_0$  is directed from the center of the earth toward the orbital station. The position of the body relative to the orbital station is determined by the radius vector  $\mathbf{r}$ , and relative to the center of the earth by a vector  $\mathbf{r}'$ , that is the sum of  $\mathbf{r}_0$  and  $\mathbf{r}$ :

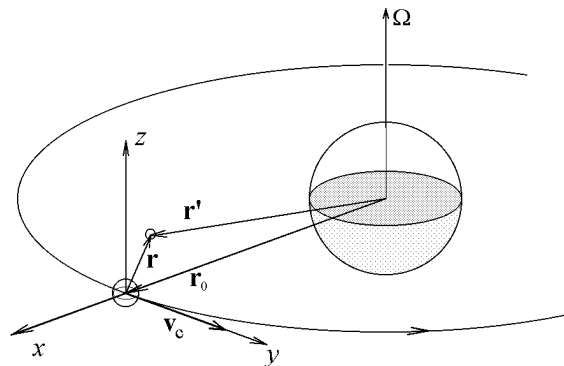


Figure 3. The frame of reference associated with the orbital station

The acceleration  $\mathbf{a}$  of the body relative to this rotating frame of reference is determined by the gravitational pull of the earth  $-GmM\mathbf{r}'/r'^3$  (here  $M$  is the mass of the earth and  $m$  is the mass of the body), and also by the centrifugal force of inertia  $-m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}')$ , and by the Coriolis force of inertia  $2m\mathbf{v} \times \boldsymbol{\Omega}$ , where  $\mathbf{v}$  is the vector of velocity of the body relative to the station. Thus,

$$\mathbf{a} = -\frac{GM}{r'^3} \mathbf{r}' - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}') + 2\mathbf{v} \times \boldsymbol{\Omega}. \quad (2)$$

Now we represent vector  $\mathbf{r}'$  in Eq. (2) as the sum  $\mathbf{r}_0 + \mathbf{r}$  (see Figure 3). To simplify the expression  $1/r'^3$  in the case of small spatial distances  $r \ll r_0$ , we calculate first the square of the vector  $\mathbf{r}'$ , and then raise the expression obtained to the power  $-3/2$ . Calculating the square of the vector  $\mathbf{r}_0 + \mathbf{r}$ , we reject the small term  $r^2$ . Leaving only the terms linear in small quantity  $r/r_0$ , for the first term in the right-hand side of Eq. (2) we obtain:

$$\frac{GM}{r'^3} \approx \frac{GM}{r_0^3} \left(1 + 2 \frac{\mathbf{r}_0 \cdot \mathbf{r}}{r_0^2}\right)^{-3/2} \approx \Omega^2 \left(1 - 3 \frac{\mathbf{r}_0 \cdot \mathbf{r}}{r_0^2}\right). \quad (3)$$

We have taken into account here that the square of the angular velocity  $\boldsymbol{\Omega}$  of the station in its orbital motion equals  $GM/r_0^3$ . Substituting Eq. (3) into Eq. (2), we obtain the following expression for the relative acceleration of the body, valid up to the terms linear in  $r/r_0$ :

$$\mathbf{a} = -\Omega^2(\mathbf{r}_0 + \mathbf{r}) + 3\Omega^2 \frac{\mathbf{r}_0 \cdot \mathbf{r}}{r_0^2} \mathbf{r}_0 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_0) - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\mathbf{v} \times \boldsymbol{\Omega}. \quad (4)$$

In Eq. (4), the main term of the acceleration caused by the gravitational pull of the earth ( $-\Omega^2\mathbf{r}_0$ ) is balanced by the main term of the acceleration caused by the centrifugal force of inertia, namely by the term  $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_0)$ . This relationship is clearly seen from Figure 3. The balancing of the gravitational force by the force of inertia is the sense of weightlessness experienced by astronauts on the orbital station. The balancing is complete for a body located at the origin of the non-inertial frame of reference associated with the station.

For a body at some distance  $\mathbf{r}$  from the origin, only the main term of the gravitational acceleration (of order zero in  $r/r_0$ ) is balanced by the force of inertia. The remaining terms of the gravitational acceleration (linear in  $r/r_0$ ) in Eq. (4), together with the terms of the same order of magnitude in the acceleration caused by the centrifugal force of inertia, and with the acceleration produced by Coriolis force (which depends on the relative velocity  $\mathbf{v}$ ), give the desired differential equations of motion of the ejected body relative to the orbital station.

Projections of the vectors in Eq. (4) onto the axes of the reference frame associated with the station give the following system of differential equations describing approximately the relative motion:

$$\begin{aligned} \ddot{x} &= 3\Omega^2 x + 2\Omega \dot{y}, \\ \ddot{y} &= -2\Omega \dot{x}, \\ \ddot{z} &= -\Omega^2 z. \end{aligned} \quad (5)$$

Here  $x$ ,  $y$ , and  $z$  are the components of radius vector  $\mathbf{r}$  that determines the position of the body relative to the station, and  $\dot{x}$ ,  $\dot{y}$ , and  $\dot{z}$  are the components of the relative velocity.

The motion of the body ejected from the station starts from the origin of the non-inertial reference frame. Therefore, in all cases under consideration  $x(0) = 0$ ,  $y(0) = 0$ ,  $z(0) = 0$ . Next we find particular solutions of Eqs. (5) for different directions of the initial velocity  $\Delta\mathbf{v}$  of the body relative to the station.

1. The body is ejected from the station in a direction perpendicular to the plane of the orbit; that is,  $\dot{x}(0) = 0$ ,  $\dot{y}(0) = 0$ , and  $\dot{z}(0) = \Delta v$ . For these initial conditions, the particular solution of Eqs. (5) describes rectilinear oscillatory motion along the  $z$ -axis:  $x(t) = 0$ ,  $y(t) = 0$ , and  $z(t) = (\Delta v/\Omega) \sin \Omega t$ . That is, the body moves sinusoidally relative to the station along the  $z$ -axis, with the station at the center of the motion and with a period  $T = 2\pi/\Omega$ . This period is equal to the period of revolution of the station along its circular orbit. At every quarter of this revolution the body is at its maximum distance from the station:  $l = \Delta v/\Omega = r_0(\Delta v/v_c)$ .

As an example we let the height of the circular orbit of the station be  $h = 0.1 R \approx 640$  km (the radius of the orbit  $r_0 \approx 7$  thousand kilometers, and the period of revolution  $T \approx 98$  minutes), and the relative initial velocity of the body be  $\Delta v = 15$  m/s, which is approximately 0.2% of the orbital velocity  $v_c = 7.5$  km/s. In this case the maximal distance  $l$  of the body from the station is approximately 14 km.

This result can be easily understood in terms of the motion of the body in the geocentric frame of reference. In this case the body moves along an almost circular orbit that differs from the orbit of the station only by a slightly different orientation of its plane. The angle between these planes equals the (small) angle  $\alpha$  between vector  $\mathbf{v}_c$  of the circular velocity of the station and vector  $\mathbf{v}_c + \Delta\mathbf{v}$  of the geocentric initial velocity of the body. Here  $\alpha \approx \Delta v/v_c \approx 2 \cdot 10^{-3}$ . Making revolutions along their orbits with practically equal periods, the body and the station meet after every half-revolution at the points of intersection of their orbits. In the meantime, they recede from one another through a maximal distance  $l = r_0\alpha = r_0(\Delta v/v_c)$ .

2. The body is ejected from the station in the radial direction, e.g., downward (toward the earth):  $\dot{x}(0) = -\Delta v$ ,  $\dot{y}(0) = 0$ , and  $\dot{z}(0) = 0$ . For these initial conditions, the second of Eqs. (5) gives  $\dot{y} = -2\Omega x$ . Substituting  $\dot{y}$  into the first of Eqs. (5), we obtain  $\ddot{x} = -\Omega^2 x$ , whence  $x(t) = -(\Delta v/\Omega) \sin \Omega t$ . Now from the equation  $\dot{y} = -2\Omega x$  at the initial condition  $y(0) = 0$  we find  $y(t) = -2(\Delta v/\Omega)(\cos \Omega t - 1)$ .

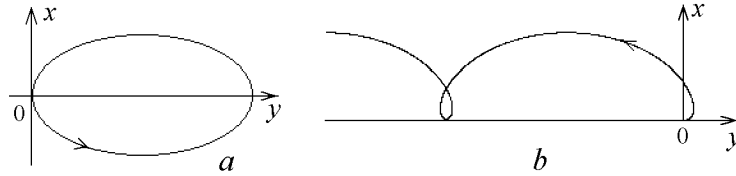
Since  $z(t) = 0$ , the relative motion of the body described by these equations occurs in the  $x, y$ -plane. To find an explicit expression for the shape of the trajectory, we eliminate the time  $t$  from the equations for  $x(t)$  and  $y(t)$ . We obtain

$$\frac{x^2}{l^2} + \frac{(y - 2l)^2}{(2l)^2} = 1. \quad (6)$$

Here we again make use of the notation  $l = \Delta v/\Omega = r_0(\Delta v/v_c)$  for the characteristic distance. From Eq. (6) we see that in the reference frame associated with the orbital station, the body moves along an ellipse (Figure 4, *a*), whose semiminor axis equals  $l = r_0(\Delta v/v_c)$  (about 14 km for  $\Delta v = 15$  m/s), and whose semimajor axis is twice as long and is oriented along the orbital velocity of the station.

Thus the body ejected downward toward the earth does in fact at first move vertically down, relative to the station. This is what the astronauts on the station see when they watch the motion of the body with the naked eye. However as the body continues moving, it gradually deviates from its almost vertical initial trajectory. After a half-revolution of the station in its orbit, the body is located in front of the station a distance of  $4l$  (about 56 kilometers using the earlier numerical example). After the





**Figure 4.** Trajectories of a body ejected from an orbital station (a) toward the earth and (b) forward in the direction of the orbital motion.

next half-revolution, the body returns to the station from the opposite side, that is, from above (see Figure 4, a).

The deviation of the body from its initial downward path becomes considerable only after the body has moved several kilometers away from the station. Very likely the astronauts will have lost sight of a small body by then. During the period they can keep the body in sight, it moves towards the ground; when it reappears to the unaided eye, it does so from above!

Such a periodic motion of the body relative to the station also can be easily explained from the point of view of a geocentric observer. Such an explanation is given above (section 1), together with a description of the corresponding simulation experiment.

3. If the body is ejected from the station parallel or antiparallel to the orbital velocity  $\mathbf{v}_c$ , its period of revolution along its elliptical orbit is no longer equal to the period of revolution of the station. A *secular term* that steadily increases with time appears in the equations for its relative motion.

For the initial conditions  $\dot{x}(0) = 0$ ,  $\dot{y}(0) = \Delta v$ , and  $\dot{z}(0) = 0$  the particular solution of the system of Eqs. (5) is:

$$\begin{aligned} x(t) &= 2l(1 - \cos \Omega t), \\ y(t) &= l(-3\Omega t + 4 \sin \Omega t), \\ z(t) &= 0. \end{aligned} \tag{7}$$

Here the previous notation  $l = r_0(\Delta v/v_c)$  for the characteristic distance is used. In the radial direction (along the local vertical line) the relative motion is again periodic. However, in the direction tangential to the circular orbit (along  $y$ -axis), simultaneously with periodic oscillations, a steady (monotone) variation of the coordinate occurs with an average velocity of  $-3\Omega l = -3\Delta v$ .

The trajectory of this motion is shown in Figure 4, b. (The scale along the ordinate axis is exaggerated.) At first the body, ejected forward from the station, does in fact move forward relative to the station, in the direction of the relative initial velocity imparted to the body. But soon the body turns upward and begins a retrograde motion. After a time  $T$  that equals the period of revolution of the station, the body returns to its initial height above the earth, but behind the station by a distance  $6\pi l$  (about 265 km for a quite small initial velocity  $\Delta v = 15$  m/s used in the earlier examples). After the next interval  $T$  the distance of the body behind the station doubles, and so on.

A diagram of the relative motion in this case (obtained in the simulation experiment for a rather large value of the initial velocity of the body) is shown on the right-hand side of Figure 2.

**Acknowledgement**

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