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Peculiarities of simulations in nonlinear systems

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Simple deterministic nonlinear mechanical systems such as the parametrically or directly driven pendula exhibit a surprisingly large variety of stable periodic and chaotic motions. During the last decades, much insight has been gained into the nature of these phenomena. Although the apparent contradiction between determinism and randomness is now rather well understood mathematically, perhaps a better introduction to chaos is a plain demonstration of a simple mechanical system in action. In order to observe chaotic behavior, which is possible only in nonlinear systems, numerical simulation is an essential tool. The principal aim of this contribution is to present some part of a vast collection of various simple and very complicated, sometimes counterintuitive, regular (periodic, phase-locked to the external drive) and chaotic (pseudo random) motions discovered recently in computer simulations of simple physical systems.

An ordinary rigid planar pendulum whose axis is driven sinusoidally in the vertical direction is a paradigm of contemporary nonlinear dynamics. This mechanical system is also interesting because the differential equation of the pendulum is frequently encountered in various problems of modern physics. Mechanical analogues of physical systems allow a direct visualization and thus can be very useful in gaining an intuitive understanding of complex phenomena (see, for example, [1]). A mode of chaotic motion of such a pendulum is shown in Figure 1.

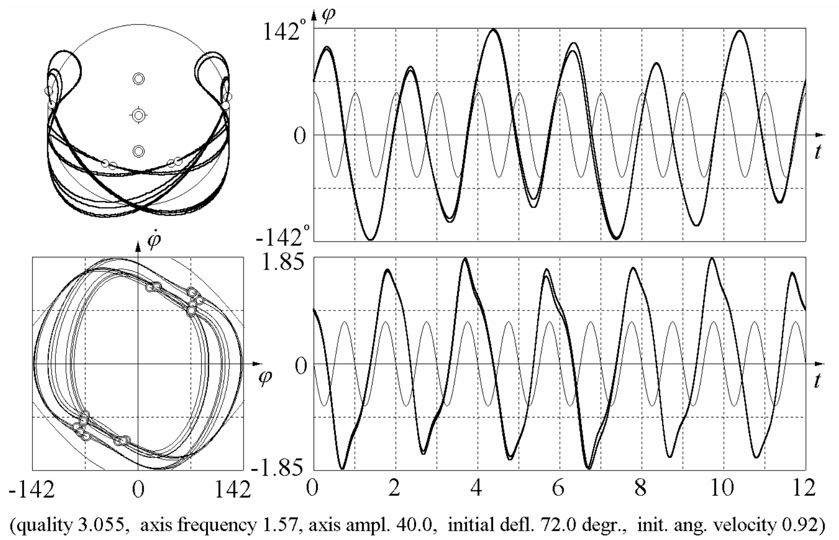

Figure 1. Chaotic oscillations of the parametrically driven pendulum. The spatial trajectory of the pendulum's bob and the phase trajectory with Poincare sections (left), time-dependent plots (right).

This chaotic motion is purely oscillatory, and nearly (but not exactly) repeats itself after each six driving periods. The six bands of Poincare sections make two groups of three isolated islands each. The representing point visits these groups in alternation. It also visits the islands of each group in a quite definite order, but within each island the points continue to bounce from one place to another without any apparent order.

The six-band chaotic attractor has a rather extended (and very complicated in shape) domain of attraction in the phase plane of initial states. Nevertheless, at these values of the control parameters the system exhibits multiple asymptotic states: this chaotic attractor coexists with several periodic regimes. Figure 2 illustrates a regular steady-state oscillation whose period (as well as the period of its fundamental harmonic) equals 4 driving cycles. However, the period of the harmonic component that dominates the spectrum equals two driving periods because the general character of oscillations reminds the principal parametric resonance. This period-4 mode is excited if the initial state belongs to a different region of the phase plane than the domain of attraction of the chaotic regime. Other initial conditions (at the same parameters) bring the pendulum to familiar period-2 steady-state parametric oscillations.

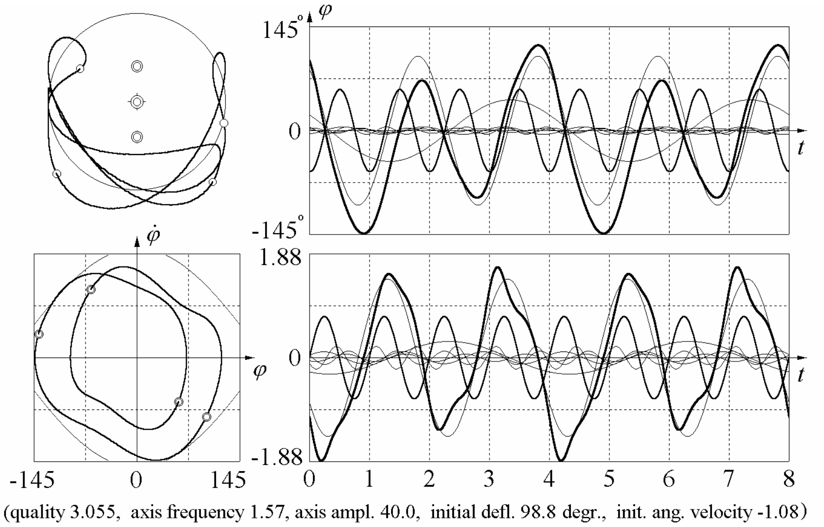

Figure 2. Regular oscillations occurring at the same values of parameters as the chaotic oscillations shown in Fig.1. Their period equals 4 driving cycles. The graphs of oscillations are shown together with their harmonic components.

The behavior of the parametrically excited pendulum is much richer in various modes than we can expect for such a simple physical system relying on our intuition. Other numerous kinds of its complicated regular and chaotic motions are simulated in a computer program available in the web [2]. Those include, in particular, subharmonic resonances (multiple-periodic stationary states), for which a clear physical explanation is suggested in [3] and an approximate quantitative theory is developed. Also subharmonic resonances of fractional orders are described and explained for the first time in [3]. However, many of the discovered modes (see [2]) are still waiting a plausible physical explanation.

A common way to investigate the routes to chaos in a nonlinear system consists of a slow variation of some parameter at fixed values of all other parameters. If we start from some regular (periodic) mode and then, without interrupting the motion, let vary gradually, for example, the driving frequency or amplitude (or the damping factor), we can observe intriguing sequences of bifurcations, including the symmetry breaking phenomena, period doubling cascades, etc., which can bring the system to some chaotic regime described by a strange attractor in the phase plane. However, numerous regular and chaotic regimes do exist, which cannot be reached in this way. Even if the required values of all the parameters are encountered during the sweeping, this may be insufficient for excitation of the mode in question, because the mode may be characterized by some (narrow) domain of attraction in the phase space of initial states. As a rule, a nonlinear system can live in multiple different stable modes at the same values of all the controlling parameters (multistability). In other words, different types of stable response coexist. Which one of these asymptotic modes is realized in a certain experiment, crucially depends on the starting conditions.

References

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2. Eugene Butikov. *The parametrically driven pendulum* (the simulation program) http://www.ifmo.ru/butikov/Inverted.html
3. Eugene Butikov. *Subharmonic resonances of the parametrically driven pendulum*, Journ. Phys. A: Math. Gen. **35**, 6209-6231, 2002