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Misconceptions about the energy of waves in a strained string

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Abstract
The localization of the elastic potential energy associated with transverse and longitudinal waves in a stretched string is discussed. Some misunderstandings about different expressions for the density of potential energy encountered in the literature are clarified. The widespread opinion regarding the inherent ambiguity of the density of elastic potential energy is criticized.

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(Some figures may appear in color only in the online journal)

1. Introduction

A serious confusion exists in the minds of scientists and in the literature concerning the density of potential energy associated with waves in a strained string. Some authors claim that the energy density in a string cannot be uniquely specified. Different expressions for the density of elastic potential energy have been suggested and discussed in several recently published papers [1, 2, 4]. The situation is certainly inadequate because the paradigm of classical mechanics leaves no room for such ambiguities and uncertainties.

This confusion originates from the calculation of the potential energy stored in a string in a well-known classic textbook of Morse and Feshbach [5]. Unfortunately, the results of this calculation are accompanied in [5] by erroneous comments—one can encounter misconceptions even in such reputable books. Comparing two different expressions for the elastic potential energy, Morse and Feshbach come to the conclusion: 'The potential energy of a string element is not unique, because the question of the energy of the endpoints of the element under consideration cannot be uniquely determined.' However, this invalid conclusion essentially does not follow from the calculations presented in [5], as we show further in this paper.

From the general perspective, the interpretation suggested in the above citation is unsatisfactory, because it is based on the notion of the energy stored in a point. Indeed, the elastic potential energy stored in a medium depends on how the medium is deformed with respect to its equilibrium state. It makes sense to consider deformation of a segment (even if infinitesimal) of the string, but not deformation of a point. A material point is characterized by its spatial position, but it has no form or dimensions. Deformation of a material point does not make sense. When the string is regarded as a continuous system, that is, as a system with distributed parameters (in contrast to a system with lumped parameters), a finite energy (potential or kinetic) can be stored only in a *segment* of a string, not in a *point*. Therefore the notion of 'the energy of the endpoints' can hardly have any physical meaning for a continuous string.

Specifically, the calculation by Morse and Feshbach [5] is applicable only to the potential energy of the *entire string*, but not to the potential energy of its arbitrary segment. This calculation cannot say anything about the spatial distribution of the elastic potential energy along the string. Below we show how this calculation can be modified in order to obtain the proper unambiguous expression for the true density of the potential energy associated with a wave in a string.

2. The potential energy of a transversely distorted string

The elastic potential energy stored in a string depends uniquely on the instantaneous shape of the string. For simplicity, we consider planar distortions, which can be described by two scalar quantities: momentary longitudinal displacement $\xi(x,t)$ of a string point whose equilibrium coordinate is x and displacement $\psi(x,t)$ of this point in the

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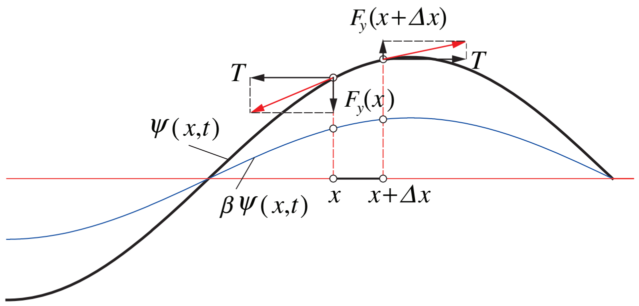

Figure 1. The string shape $\psi(x, t)$ in a transverse wave at some instant t and the forces exerted on the segment Δx by its neighbors.

transverse direction. In this section, we concentrate on the contribution of transverse distortions $\psi(x, t)$ to the elastic potential energy of the string.

The momentary transverse displacements of the string points in a wave at a certain time instant t are shown in figure 1. Let us consider an elementary string segment, which in the absence of a wave lies between x and $x + \Delta x$. In an undisturbed stretched string each segment already stores some elastic potential energy. However, we are interested here only in the additional potential energy associated with a disturbance caused by the wave. The elementary treatments in most textbooks (see, e.g., [6, 7]) typically assume that the additional potential energy of the string element which is disturbed by a transverse wave of small amplitude is given approximately by the following expression:

$$\Delta E_{\text{pot}} = \frac{1}{2} T \left(\frac{\partial \psi(x, t)}{\partial x} \right)^2 \Delta x. \tag{1}$$

This expression is usually treated as the work done by the approximately constant tension T in additional stretching of the string element through $\frac{1}{2}(\partial\psi/\partial x)^2 \Delta x$ as the element is distorted and displaced transversely from the undisturbed position into the current position with the left end of the segment Δx located at $(x, \psi(x, t))$ and the right end—at $(x + \Delta x, \psi(x + \Delta x, t))$.

However, it seems natural that one can obtain the additional potential energy of the string segment Δx also by calculating the work done by transverse forces, directed along the displacement of the segment. This alternative approach is used by Morse and Feshbach [5].

In order to better understand what is actually calculated in [5], next we consider the transverse forces in more detail. When there is no wave disturbance, the forces exerted on the segment Δx by its left and right neighbors have opposite directions, and their magnitudes are equal to the tension T of the undisturbed string. In a wave, the left end of the segment at time t is displaced in the transverse y-direction through distance $\psi(x, t)$, and its right end—through distance $\psi(x + \Delta x, t)$. The elastic forces exerted on a segment of a perfectly flexible string by its neighbors are directed tangentially to the string, so that at the left end the force makes an angle with the x-direction whose tangent equals $-\partial\psi(x, t)/\partial x$, and at the right end equals $\partial\psi(x + \Delta x, t)/\partial x$.

In a purely transverse wave the segment does not move in the longitudinal direction; hence, the x-components of the left and right forces are always equal to the tension T. Therefore the transverse force $F_y(x)$ exerted on the left end is $-T\partial\psi(x, t)/\partial x$ and the force $F_y(x + \Delta x)$ exerted on the right end is $T\partial\psi(x + \Delta x, t)/\partial x$. Hence, the net force ΔF_y exerted on the elementary segment Δx by its left and right neighbors at the time t is proportional to the second spatial derivative of $\psi(x, t)$:

$$\Delta F_y = T \left[\frac{\partial \psi(x + \Delta x, t)}{\partial x} - \frac{\partial \psi(x, t)}{\partial x} \right] \approx T \frac{\partial^2 \psi(x, t)}{\partial x^2} \Delta x. \tag{2}$$

In a vibrating string (a string with a wave), this transverse net force ΔF_y imparts acceleration $\partial^2\psi/\partial t^2$ to the string element Δx, whose mass is equal to $\rho_l \Delta x$ (ρ_l is the linear density of the string, that is, the mass per unit length of the strained but undisturbed string). According to Newton's second law, we should equate the net force given by equation (2) to $\rho_l \Delta x (\partial^2\psi/\partial t^2)$. This produces the standard wave equation with $v_{\text{T}} = \sqrt{T/\rho_l}$—the speed of transverse waves. Traveling and standing waves equally satisfy this equation. We emphasize that the work of the transverse elastic force ΔF_y, equation (2), exerted on the element Δx of the oscillating string by its neighbors changes the *total* energy of the element, that is, both potential and kinetic energies (not only the potential energy).

The kinetic energy of a given string segment Δx in a transverse wave is equal to $\frac{1}{2}\rho_l \Delta x (\partial\psi/\partial t)^2$. To calculate the additional potential energy associated with a given momentary shape $\psi(x, t)$ of the string on the basis of work considerations, Morse and Feshbach [5] assumed that transition of the string from its equilibrium shape $\psi = 0$ to the given shape $\psi(x, t)$ is executed quasi-statically through a sequence of intermediate shapes.[1]

This sequence of shapes can be described by a function $\beta\psi(x, t)$ (see figure 1). For the initial (undisturbed) state $\psi = 0$ parameter β is equal to zero; all intermediate shapes correspond to values of β changing from 0 to 1; the final shape $\psi(x, t)$ is obtained at $\beta = 1$. In an intermediate state the transverse elastic force exerted on the segment by its neighbors is equal to $\beta \Delta F_y$ with ΔF_y given by equation (2).

To provide a quasi-static transition of the string from $\psi = 0$ to $\psi(x, t)$, the net elastic force (2) from the neighbors must be balanced (in all intermediate configurations) by an equal and opposite external force that should be exerted on the string element Δx. In an intermediate configuration characterized by some value of parameter β, this external force is equal to $-\beta \Delta F_y$, where ΔF_y is given by equation (2). The work ΔW done by this force while the segment Δx moves to its position $\psi(x, t)$ in the final configuration of the string is

$$\Delta W = -\int_0^1 \beta T \left(\frac{\partial^2 \psi}{\partial x^2} \right) \psi \, \mathrm{d}\beta \cdot \Delta x = -\frac{1}{2} T \psi \left(\frac{\partial^2 \psi}{\partial x^2} \right) \Delta x. \tag{3}$$

[1] We note an erroneous statement in [8]: '…the work done on a string element to bring it to a particular configuration must be the same whether the distortion of the string was done quasi-statically or otherwise. Quasi-staticity is indeed assumed in the derivation for simplicity, yet the result cannot depend on quasi-staticity.' This is another misunderstanding about the potential energy. As already mentioned above, in an oscillating string the work done by unbalanced transverse forces, equation (2), is equal to the change in the *total* energy of the string element.

However, we claim that this work ΔW cannot be treated as the potential energy stored in the segment Δx, and equation (3) does not describe the actual distribution of potential energy in the distorted string. Indeed, it is impossible to move the segment Δx to the final position $\psi(x,t)$ without moving its neighbors. The required equilibrium of the entire distorted string in all intermediate positions can be provided only by exerting appropriate external forces simultaneously on all the elements of the string. Hence only the total amount of work done by all these external forces gives the potential energy stored in the entire distorted string. This energy is calculated by integrating expression (3) along the whole string (between fixed endpoints $x=a$ and $x=b$):

$$E_{\text{pot}} = -\frac{1}{2}T\int_a^b \psi\left(\frac{\partial^2\psi}{\partial x^2}\right)\mathrm{d}x. \tag{4}$$

We note that the total amount of work done by elastic forces during the quasi-static transition from $\psi=0$ to $\psi(x,t)$ is zero, because the forces of interaction between adjoining segments of the string are equal and opposite. The potential energy stored in the whole string, equation (4), is equal to the work of *external* forces that are needed to balance the elastic forces during the quasi-static transition of the string from the undisturbed state $\psi=0$ to $\psi(x,t)$. The expression (4) was obtained by Morse and Feshbach [5] for the potential energy associated with a wave in a string. The existence of two different expressions, (1) and (3), which, being integrated along the string, give the same value for the potential energy, was interpreted in [5] as an impossibility to define uniquely the density of potential energy in a wave. Our claim is that although equation (4) is certainly correct, the integrand in (4), which is given by equation (3), cannot be regarded as the true linear density of the elastic potential energy. We emphasize that the approach used in [5] allows us to calculate only the total amount of potential energy stored in the entire string.

The true expression for the linear density should unambiguously describe actual localization of elastic potential energy in the string. To obtain this expression from work considerations, instead of calculating the work needed to *move* the given segment from $\psi=0$ to $\psi(x,t)$ as is done above and in [5], we should rather calculate the work of external forces needed to quasi-statically produce the required final *distortion* of this segment through all intermediate distortions. Indeed, the potential energy stored in an individual segment of the string depends on distortion of the element, but not on its absolute position. To perform this calculation, let us consider the left end of the segment Δx to be immovable, while its right end is moved transversely through intermediate positions over the distance $(\partial\psi/\partial x)\Delta x$ (see figure 1). To characterize intermediate states of the segment Δx in this transition, let us introduce, following the calculation by Morse and Feshbach [5], some parameter β that varies from 0 to 1. In an intermediate position, when the right end of this segment occurs at the point $(x+\Delta x,\ \beta\Delta x(\partial\psi/\partial x))$, we should exert on the right end a transverse external force $\beta T(\partial\psi/\partial x)$, together with the longitudinal force T, in order to provide the equilibrium of the segment. The work done by this external force is equal to the potential energy stored in the segment Δx:

$$\Delta E_{\text{pot}} = \int_0^1 \beta T\left(\frac{\partial\psi}{\partial x}\right)\left(\frac{\partial\psi}{\partial x}\right)\mathrm{d}\beta\cdot\Delta x = \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2\Delta x. \tag{5}$$

We find that this calculation gives just the commonly used expression (1) for potential energy that can be found in most textbooks. The true spatial distribution of the additional elastic potential energy associated with a purely transverse distortion of a string to instantaneous shape $\psi(x,t)$ is uniquely described by the linear density $\varepsilon_{\text{pot}}=\frac{1}{2}T(\partial\psi/\partial x)^2$, which is proportional to the square of the transverse fractional distortion. For the entire string, this expression certainly gives the same value E_{pot} of the stored potential energy as the above-cited calculation by Morse and Feshbach [5]. Indeed, from equation (4), we obtain through integration by parts:

$$E_{\text{pot}} = \int_a^b \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2\mathrm{d}x - \frac{1}{2}T\left[\psi\left(\frac{\partial\psi}{\partial x}\right)\right]_a^b. \tag{6}$$

The integrand in the first term is exactly the commonly used expression (1) for the linear density of potential energy. When the string is of finite length and a standing wave is excited, the boundary term in equation (6) vanishes for either fixed or free endpoint boundary conditions, because at points $x=a$ and $x=b$ either $\psi=0$ (nodes) or $\partial\psi/\partial x=0$ (antinodes). When the string is infinitely long, the boundary term also vanishes for any disturbance of finite length and durance.

The above discussion refers only to potential energy of the string associated with purely transverse distortions of the string.[2] We claim that there is no inherent ambiguity in the potential energy density associated with waves in a string, contrary to the misleading discussions one can encounter in the literature since the textbook of Morse and Feshbach [5]. For example, Rowland [1, 3] claims that the ambiguity in the potential energy density arises when only the transverse motion of string elements is considered and that this ambiguity is removed only by taking into account the longitudinal motion of elements of the string ('…the difference is merely an artefact of making the assumption that longitudinal motion can be neglected'—a quotation from [3]).

Any longitudinal distortion described by some function $\xi(x,t)$ gives an *independent* unambiguous contribution to the potential energy. This contribution is discussed below in section 4. We emphasize again that the standard expression (1) refers only to a purely transverse distortion whose shape $\psi(x,t)$ uniquely defines the corresponding contribution to the potential energy of the string. The existence of an unambiguous expression, equation (1), for the potential energy density associated with transverse distortions is not related, contrary to the statement in [3], to possible generation

[2] Purely transverse waves in a stretched string can exist if longitudinal and transverse waves are characterized by equal velocities. This condition holds for a slinky spring, which is often used as a convenient tool for lecture demonstrations. In the general case, transverse waves excited in a stretched string produce additional small longitudinal distortions due to nonlinear effects. Rowland [3] has shown that these small longitudinal distortions can make a contribution to the potential energy density of the same order of magnitude as the original transverse distortions. This means that generally it may be necessary to take these nonlinear effects into account in calculating the true density of potential energy. If the speed of longitudinal waves is much greater than the speed of transverse waves, the elastic potential energy in a wave is almost uniformly distributed along the string by virtue of these additional longitudinal distortions.

of the longitudinal distortions caused by the internal forces acting between adjoining segments of a transversely distorted string.

3. Energy transformations in a transverse wave

For a purely transverse traveling wave of an arbitrary shape $\psi(x,t) = f(x - v_\mathrm{T}t)$, equation (1) shows that the linear densities of kinetic and potential energies are equal to one another at a spatial point x at a time instant t; they rise and fall together. In particular, for a sinusoidal wave $\psi(x,t) = A\sin(k_\mathrm{T}x - \omega t)$ (here $k_\mathrm{T} = \omega/v_\mathrm{T}$), both ε_kin and ε_pot oscillate with frequency 2ω, reaching simultaneously their minimum (zero) values at crests and troughs and maximum values $\frac{1}{2}\rho_l\omega^2A^2$ (equal for both) at points of zero displacement $\psi(x,t) = 0$. Clear qualitative and quantitative descriptions of the energy transformations in a transverse sinusoidal traveling wave can be found in many standard textbooks.

The commonly accepted expression (1) for the potential energy density ε_pot is consistent with the conservation of energy. Indeed, the power $P(x,t)$ transmitted by a transverse wave through some point of the string can be calculated as the product of the transverse force $-T(\partial\psi/\partial x)$ exerted at a point x on the adjoining string element, and the velocity $(\partial\psi/\partial t)$ of this point. For a sinusoidal traveling wave this yields:

$$P(x,t) = -T\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t} = \frac{1}{2}\rho_l\omega^2A^2v_\mathrm{T}[1+\cos 2(k_\mathrm{T}x - \omega t)]. \tag{7}$$

We see that $P(x,t)$ equals the total energy density $(\varepsilon_\mathrm{kin} + \varepsilon_\mathrm{pot})$ times the wave speed v_T. The momentary value of $P(x,t)$ oscillates with the frequency 2ω between zero and the maximum value $\rho_l\omega^2A^2v_\mathrm{T}$. This unidirectional energy flow through any point x of the string is constant only on the time average: $\langle P(x,t)\rangle = \frac{1}{2}TA^2\omega k_\mathrm{T} = \frac{1}{2}\rho_l\omega^2A^2v_\mathrm{T}$.

For a standing wave described, say, by the wavefunction $A\sin(k_\mathrm{T}x)\sin(\omega t)$, the densities of kinetic and potential energies are given by the following expressions:

$$\varepsilon_\mathrm{kin} = \tfrac{1}{2}\rho_l A^2\omega^2\sin^2(k_\mathrm{T}x)\cos^2(\omega t), \tag{8}$$

$$\varepsilon_\mathrm{pot} = \tfrac{1}{2}\rho_l A^2\omega^2\cos^2(k_\mathrm{T}x)\sin^2(\omega t). \tag{9}$$

At the nodes, $k_\mathrm{T}x = n\pi$ $(n = 0, \pm1, \pm2, \ldots)$ and ε_kin is always zero, while ε_pot oscillates between zero and the maximum value $\frac{1}{2}\rho_l\omega^2A^2$ with the frequency 2ω. At the antinodes, $\cos k_\mathrm{T}x = 0$ and ε_pot is always zero, while ε_kin oscillates between zero and the same maximum value $\frac{1}{2}\rho_l\omega^2A^2$. The density of the total mechanical energy in the string $\varepsilon_\mathrm{kin} + \varepsilon_\mathrm{pot}$ also oscillates with time with the frequency 2ω and with an amplitude that is position dependent. (The amplitude has maximum values at nodes and antinodes and is equal to zero half-way between them). The energy flow $P(x,t)$ in the standing wave

$$\begin{aligned} P(x,t) &= -T\frac{\partial\psi(x,t)}{\partial x}\frac{\partial\psi(x,t)}{\partial t} \\ &= -\frac{1}{4}\rho_l\omega^2A^2v_\mathrm{T}\sin(2k_\mathrm{T}x)\sin(2\omega t) \end{aligned} \tag{10}$$

is always zero at nodes and antinodes, where $\sin(2k_\mathrm{T}x) = 0$. In particular, the flow is equal to zero at the endpoints of the string: the energy of the whole oscillating string is conserved. However, for all points of the string between a node and the adjoining antinodes this is true only for the time average: $\langle P(x,t)\rangle = 0$ over an integer number of half-periods. During a quarter period the energy flow is directed from nodes to antinodes, and during the next quarter period its direction is reversed. A quantitative description of energy transformations in a standing wave can be found in [9].

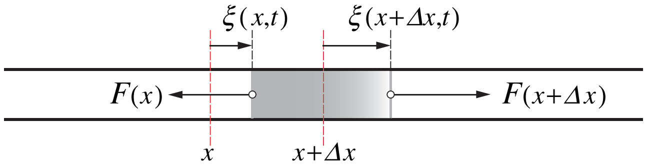

Figure 2. Displacement and distortion of a string segment Δx caused by a longitudinal wave $\xi(x,t)$ at some instant t and the forces exerted on the edges of the segment.

Nevertheless, one can encounter in the literature misconceptions concerning the energy transfer in a standing wave. For example, the author of [2] writes: 'Unlike the case of a traveling wave, in a standing wave there is no energy transfer, and the total mechanical energy of each string element is expected to be stationary.' This is certainly an erroneous statement. Adjoining elements of the string interact and exchange energy even in the standing wave. In a standing wave the mechanical energy of an individual string element is not conserved: the energy is obviously transferred back and forth between nodes and antinodes. Indeed, at the moment when the oscillating string passes through its equilibrium, the string energy is wholly kinetic and is localized near the antinodes. Conversely, a quarter period later the string energy is wholly potential and is localized near the nodes. This means that in the meantime (during this quarter period), the energy of the standing wave travels from antinodes toward nodes transforming simultaneously from kinetic energy to potential energy. During the next quarter period, the string energy is transferred back from nodes to antinodes and transformed simultaneously from potential energy to kinetic energy.

4. Potential energy of a longitudinally distorted string

Next we calculate the contribution of longitudinal distortions to the potential energy of a strained string. This issue is important not only for the case of purely longitudinal waves, or when transverse and longitudinal waves are simultaneously excited by the source. Even when only transverse waves are excited in a strained string, small longitudinal distortions may appear by virtue of nonlinear effects (see footnote 2). These distortions can make a contribution to the potential energy density of the same order of magnitude as the original transverse distortions [3].

Let the longitudinal distortion of a string caused by a wave be described by a function $\xi(x,t)$. Then the left edge of some segment Δx at time instant t is displaced from its undisturbed position x through distance $\xi(x,t)$, and the right edge—through distance $\xi(x+\Delta x,t)$, as shown in figure 2.

The elastic force $F_x(x,t)$ of interaction between adjoining segments of the stretched string consists of a constant tension T and an additional part related to nonuniform distortion produced by the wave. According to Hooke's law, this additional force is proportional to the fractional distortion of an elementary segment $[\xi(x+\Delta x)-\xi(x)]/\Delta x_0$, where Δx_0 is the equilibrium length of the segment Δx in the unstretched string (at $T=0$). Thus

$$F_x(x,t) \approx T + SY\frac{\xi(x+\Delta x)-\xi(x)}{\Delta x_0}. \qquad (11)$$

Here Y is Young's modulus of the string material and S is the cross-sectional area of the string. We can express the second term in the right-hand side of equation (11) in terms of the spatial derivative of $\xi(x,t)$ by substituting into equation (11) the undisturbed length Δx_0 through Δx from $\Delta x = \Delta x_0(1+T/SY)$. Therefore we can write the following expression for the momentary elastic force $F_x(x,t)$ in a cross-section x:

$$F_x(x,t) = T + (SY+T)\frac{\partial \xi(x,t)}{\partial x}. \qquad (12)$$

Next we consider the net force ΔF_x exerted on the segment Δx by its left and right neighbors. The force exerted on the left-hand edge of the segment is given by the negative of equation (12); for the force exerted on the right-hand edge we should replace x in equation (12) by $x+\Delta x$. Therefore the net force ΔF_x exerted on the segment Δx by its neighbors is proportional to the second spatial derivative of disturbance $\xi(x,t)$:

$$\Delta F_x = (SY+T)\left[\frac{\partial \xi(x+\Delta x,t)}{\partial x} - \frac{\partial \xi(x,t)}{\partial x}\right] \approx (SY+T)\frac{\partial^2 \xi(x,t)}{\partial x^2}\Delta x. \qquad (13)$$

We note that the first term T in the local elastic force $F_x(x,t)$, equation (12), gives no contribution to the net force ΔF_x, equation (13)—this uniform tension T produces equal and opposite forces exerted on the left and right edges of the string segment Δx.

In a string with a wave the longitudinal net force ΔF_x, equation (13), imparts acceleration $\partial^2\xi/\partial t^2$ to the string element Δx. The mass Δm of this segment is equal to $\rho_l \Delta x$, where $\rho_l = m/L$ is the linear density, or $\Delta m = m\Delta x/L$, where $m=\rho S L_0$ is the total mass of the string, ρ is volume density of the string material, L_0 is the length of the unstretched string and L is the undisturbed (equilibrium) length of the string, which is stretched uniformly by the force $T = SY(L-L_0)/L_0$. By equating the net force given by (13) to $\Delta m(\partial^2\xi/\partial t^2)$, we obtain the wave equation for the longitudinal motion with the wave speed $v_L^2 = (SY+T)/\rho_l = (Y/\rho)(1+T/SY)^2 \approx Y/\rho$.

We emphasize that in the string with a wave, the work of the elastic force (13) exerted on the element Δx by its neighbors changes the *total* energy of the segment, that is, both potential and kinetic energies. The kinetic energy of the string segment Δx in a longitudinal wave is equal to $\frac{1}{2}\Delta m(\partial\xi/\partial t)^2$. To calculate from work considerations the potential energy stored in the string whose longitudinal distortion is described by some function $\xi(x,t)$, we should assume that the transformation of the string from the undisturbed state $\xi=0$ to $\xi(x,t)$ occurs quasi-statically through a sequence of intermediate configurations. Following the calculation by Morse and Feshbach [5], we introduce again some parameter β varying from 0 to 1, so that the string distortion in an intermediate configuration is given by $\beta\xi(x,t)$. During the quasi-static transformation the net elastic force exerted on the string segment Δx by its neighbors in all intermediate configurations should be balanced by an equal and opposite external force $-\beta\Delta F_x$ with ΔF_x given by equation (13). We note that at $T \ll SY$ this force is almost independent of the string tension T. The work ΔW done by this external longitudinal force while the segment Δx moves to its new position $\xi(x,t)$ is

$$\Delta W = -\int_0^1 \beta(SY+T)\left(\frac{\partial^2\xi}{\partial x^2}\right)\xi \, d\beta \cdot \Delta x = -\frac{1}{2}(SY+T)\xi\left(\frac{\partial^2\xi}{\partial x^2}\right)\Delta x. \qquad (14)$$

This work ΔW cannot be treated as the elastic potential energy stored in the particular segment Δx, because it is impossible to move the segment Δx to the final position $\xi(x,t)$ without moving its neighbors. The required equilibrium of the entire distorted string in all intermediate positions can be provided only by exerting appropriate longitudinal external forces simultaneously on all the elements of the string. Hence, only the total amount of work done by all these external forces gives the potential energy stored in the entire distorted string. This energy is calculated by integrating expression (14) along the whole string (between fixed endpoints $x=a$ and $x=b$):

$$E_{\text{pot}} = -\frac{1}{2}(SY+T)\int_a^b \xi\left(\frac{\partial^2\xi}{\partial x^2}\right) dx. \qquad (15)$$

We note again that the integrand in (15), which is given by equation (14), cannot be regarded as true linear density of the elastic potential energy stored in the longitudinally distorted string. Similarly to the case of transverse distortion, we can obtain from (15) another expression for the potential energy through integration by parts:

$$E_{\text{pot}} = \frac{1}{2}(SY+T)\int_a^b \left(\frac{\partial\xi}{\partial x}\right)^2 dx - \frac{1}{2}(SY+T)\left[\xi\left(\frac{\partial\xi}{\partial x}\right)\right]_a^b. \qquad (16)$$

When the string is of finite length and a standing wave is excited, the boundary term in the right-hand side of (16) vanishes, because at fixed endpoints $x=a$ and $x=b$ longitudinal displacement $\xi=0$. When the string is infinitely long, the boundary term also vanishes for any disturbance of finite length and durance. We recall that for the case of a transverse distortion the integrand in the similar expression (6) described the true localization of the elastic potential energy. Can we say the same about the integrand in (16) for the case of a longitudinal distortion?

To answer this question, let us try to calculate from work considerations the elastic potential energy stored in the given segment, rather than in the entire string. Instead of calculating the work needed to *move* the given segment to its disturbed position $\xi(x,t)$, we should calculate the work of external forces needed to quasi-statically produce the required

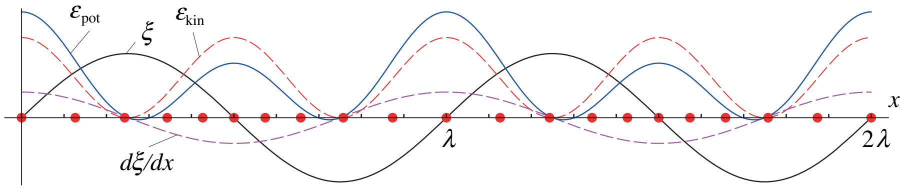

Figure 3. Instantaneous displacements of string particles in a longitudinal traveling wave $\xi(x, t)$, and the densities of potential and kinetic energies.

final *distortion* of this segment through all intermediate distortions. To do this, let us consider the left edge of the segment Δx to be immovable, while its right edge is moved through intermediate positions over the distance $\Delta x(\partial\xi/\partial x)$. To characterize intermediate states of the segment Δx in this transition, let us introduce again, following the calculation by Morse and Feshbach [5], some parameter β varying from 0 to 1. In an intermediate position, when the right edge of this segment occurs at $\beta\Delta x(\partial\xi/\partial x)$, we should exert on it, according to equation (12), an external force $T + \beta(SY + T)(\partial\xi/\partial x)$ in order to provide the equilibrium of the segment. The work done by this external force is equal to the potential energy stored in the segment Δx:

$$\Delta E_{pot} = \int_0^1 \left(T + \beta(SY+T)\frac{\partial\xi}{\partial x}\right)\frac{\partial\xi}{\partial x}\mathrm{d}\beta\cdot\Delta x$$
$$= T\left(\frac{\partial\xi}{\partial x}\right)\Delta x + \frac{1}{2}(SY+T)\left(\frac{\partial\xi}{\partial x}\right)^2\Delta x. \quad (17)$$

We note that in this calculation of the work, which is done by the external force when the left-hand end of the segment Δx is immovable, the component T of the external force (the uniform tension) executes some nonzero work, and makes a contribution to the potential energy of the segment Δx, expressed by the first term in (17).

The second term in the right-hand side of the expression (17) is proportional to the square of $(\partial\xi/\partial x)$ and coincides with the integrand of (16). For longitudinal waves in unstrained rods this is the only contribution to the potential energy: for $T = 0$ equation (17) reduces to $\Delta E_{pot} = \frac{1}{2}SY(\partial\xi/\partial x)^2\Delta x$. Therefore the localization of potential energy and transformations of energy in longitudinal waves in an unstrained elastic rod are quite similar to those in purely transverse waves on a string (see section 3).

In contrast with the expression (1) for the potential energy of a transverse distortion, which is wholly proportional to the square of $(\partial\psi/\partial x)$ (and with the potential energy of a longitudinal distortion in an unstrained rod), the first term in the right-hand side of equation (17) is proportional to the first power of $(\partial\xi/\partial x)$. Peculiarities in the energy transformations in waves on strained strings or rods under tension arise from this term, which is related with the work done by the force of uniform tension T. This work and the corresponding portion of the potential energy of segment Δx give no contribution to the energy of the entire string. The first term in equation (17) is related solely to some relocation of the elastic potential energy: for segments in which $(\partial\xi/\partial x) > 0$ this term gives a positive contribution, and vice versa. Over any traveling disturbance of finite length with $\xi = 0$ beyond the disturbance and over an entire standing wave in a string with fixed ends, this term integrates to zero. In an infinite sinusoidal wave, the mean value of this term over an integer number of wavelengths is zero. We emphasize that the potential energy described by the first term of equation (17) does not appear due to the wave disturbance: this term describes only the spatial redistribution of the energy already existing in the uniformly taut rod or string.

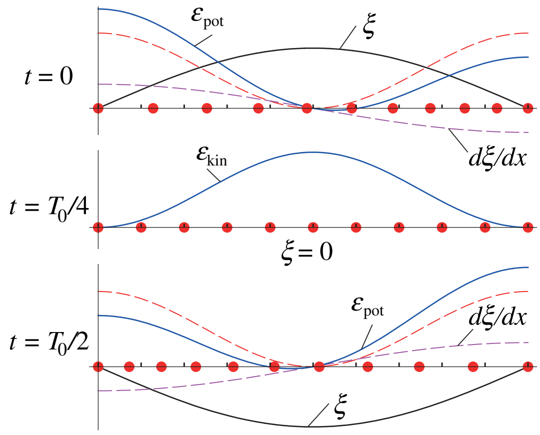

Figure 4. Instantaneous displacements of string particles in a longitudinal standing wave $\xi(x, t) = B\sin(k_L x)\cos(\omega t)$ and the densities of potential and kinetic energies.

Expression (17) for the potential energy of the wave on a strained string is compatible with the conservation of energy. The total energy flow $P(x, t)$ through arbitrary point x is given by the power of the force exerted at this point on the string by its part located to the left of x: $P(x, t)$ is equal to $-F_x(x, t)$, given by equation (12), times the longitudinal velocity $(\partial\xi/\partial t)$ of the string point x. We can easily show that the energy flow satisfies the continuity equation when the energy density $\varepsilon = \varepsilon_{kin} + \varepsilon_{pot}$ is given by the expressions: $\varepsilon_{kin} = \frac{1}{2}\rho_l(\partial\xi/\partial t)^2$, $\varepsilon_{pot} = T(\partial\xi/\partial x) + \frac{1}{2}(SY + T)(\partial\xi/\partial x)^2$. This is consistent with equation (17).

Figure 3 illustrates the redistribution of potential energy in a traveling longitudinal wave $\xi(x, t) = B\sin(k_L x - \omega t)$, $k_L = \omega/v_L$. Circles on the axis show momentary displacements of the string points from their equilibrium positions. For the portions of the string where these circles are dense, the first term in (17), proportional to $(\partial\xi/\partial x)$, is negative. Therefore at such places the potential energy density ε_{pot} is smaller than ε_{kin}. For the portions where $(\partial\xi/\partial x)$ is positive, ε_{pot} is greater than ε_{kin}. The average (over an integer number of half-wavelengths) values of ε_{pot} and ε_{kin} are equal to one another.

The potential energy (17) is transferred by the wave along the string in the direction of propagation with velocity v_L together with the kinetic energy.

Displacements of the string points and the densities of potential and kinetic energies in a standing longitudinal wave described by the function $\xi(x,t) = B\sin(k_{\mathrm{L}}x)\cos(\omega t)$ are shown in figure 4 for time moments $t = 0$, $t = T_0/4$ and $t = T_0/2$ ($T_0 = 2\pi/\omega$). At $t = T_0/4$ all string points are crossing their equilibrium positions, the potential energy $\varepsilon_{\mathrm{pot}}$ is zero, the total energy is equal to kinetic energy $\varepsilon_{\mathrm{kin}}$, which is localized near the antinodes. At $t = 0$ and $t = T_0/2$ the kinetic energy $\varepsilon_{\mathrm{kin}}$ is zero, while $\varepsilon_{\mathrm{pot}}$ reaches its maximum. The potential energy $\varepsilon_{\mathrm{pot}}$ is localized near the nodes, but due to the first term in equation (17) which is linear in fractional distortion $(\partial\xi/\partial x)$, the density $\varepsilon_{\mathrm{pot}}$ in the neighboring nodes has different values. The contributions to $\varepsilon_{\mathrm{pot}}$ of terms linear and quadratic in $(\partial\xi/\partial x)$ are shown in figure 4 by dashed lines. During a quarter period the energy travels from nodes toward antinodes, transforming simultaneously from elastic potential energy to kinetic energy. During the next quarter period these transformations reverse.

5. Concluding remarks

In this paper, we have tried to clarify misunderstandings and contradictions encountered in the literature regarding the energy of waves in strained strings. We have shown that there is no inherent ambiguity in the elastic potential energy associated with transverse waves, contrary to the widespread opinion originating from the classic textbook of Morse and Feshbach [5]. When the transverse and longitudinal distortions of the string caused by the wave are known, the localization of potential energy is uniquely defined by equations (1) and (17) for the density of potential energy. The occurrence in equation (17) of the term proportional to the first power of the longitudinal distortion explains the redistribution by the wave of potential energy already stored in the string by virtue of preliminary stretching. This unambiguous relocation of potential energy was demonstrated by considering the examples of traveling and standing longitudinal waves.

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